Quantum Information Theory

Lecture 16

1-Oct-2013

Quantum Processes & measurements

Dynamics and Measurement

Open systems
(couple to ancilla/environment)

Described by trace preserving quantum operations

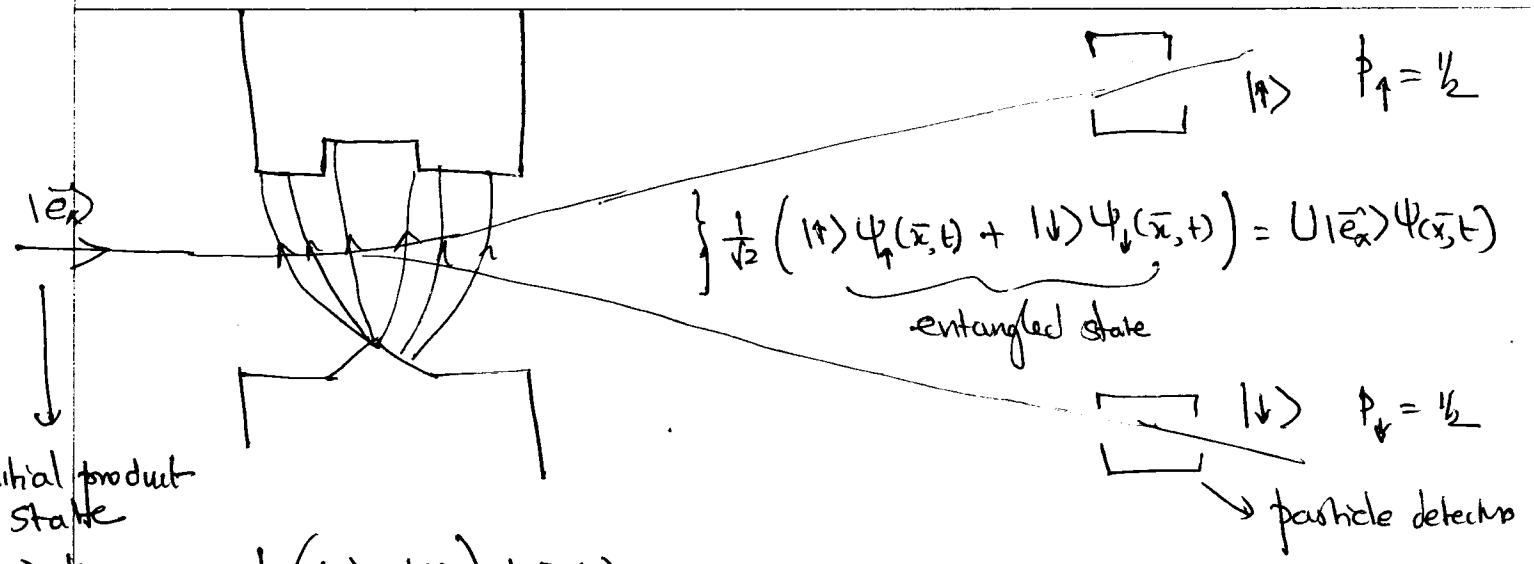
Measurement models:
Couple the system to an ancilla → "measuring apparatus"
& then observe the apparatus

Trace decreasing quantum operations

We will look at, in detail, the mathematical machinery for describing both open quantum dynamics as well as quantum measurements.

Let us first go back to the familiar Stern-Gerlach measurement and look at it in detail.

Let us assume that the incoming particles are all in the state $|\bar{e}_x\rangle$ and the axis of the inhomogenous magnetic field is along the z direction as shown in the figure below.



$$|\bar{e}_x\rangle\psi(x,t) = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\psi(\bar{x},t)$$

system ↑ ↑ apparatus

The state of the system before the particle detection is entangled between the spin & spatial degrees of freedom.

Just the two beams separating ~~out~~ out, by itself, does not constitute a measurement.

We can just as well write the intermediate state as

$$\frac{1}{\sqrt{2}}\left(|\bar{e}_x\rangle\frac{1}{\sqrt{2}}\left(\psi_\uparrow(\bar{x},t) + \psi_\downarrow(\bar{x},t)\right)\right.$$
$$\left. + |-\bar{e}_x\rangle\frac{1}{\sqrt{2}}\left(\psi_\uparrow(\bar{x},t) - \psi_\downarrow(\bar{x},t)\right)\right)$$

with each atom being in both beams at the same time. So without actually detecting a spin in one of the two beams the measurement ~~has~~ does not happen.

This setup is very similar in the way it works to a polarizing beam splitter.

Let us now get back to the familiar measurement statistics from a Stern-Gerlach system and its consequent post measurement states with a view of generalizing both.

Swabesh

Measurement statistics

(We call it statistics since finally what we are interested in are the probabilities of various outcomes & not the outcomes themselves in each instance).

$$p_\uparrow = \langle\uparrow|\rho|\uparrow\rangle = tr(\rho P_\uparrow)$$

$$p_\downarrow = \langle\downarrow|\rho|\downarrow\rangle = tr(\rho P_\downarrow)$$

Post measurement states:

$$\rho_\uparrow = \frac{P_\uparrow \rho P_\uparrow}{p_\uparrow} = |\uparrow\rangle\langle\uparrow|$$

$$\rho_\downarrow = \frac{P_\downarrow \rho P_\downarrow}{p_\downarrow} = |\downarrow\rangle\langle\downarrow|.$$

If we do not know the result of the measurement but just that a Stern-Gerlach measurement has occured, then the appropriate post measurement state is

$$\rho' = p_\uparrow \rho_\uparrow + p_\downarrow \rho_\downarrow = P_\uparrow \rho P_\uparrow + P_\downarrow \rho P_\downarrow$$

Positive operator valued measure (POVMs) [or effects]

One can generalize von Neumann type measurements into a larger class called POVMs. Mathematically a POVM is a collection of positive operators

$$E_\alpha \qquad \left(1 \geq E_\alpha \geq 0 \Rightarrow E_\alpha = \sum_j \lambda_j |e_j\rangle\langle e_j| \quad 0 \leq \lambda_j \leq 1\right)$$

Such that

$$\sum_\alpha E_\alpha = \mathbb{1}$$

Swabesh

We can make the following comparison between orthogonal projector (OP) & POVMs.

Orthogonal projector (OP)	POVM $\{E_\alpha\}$
Observable $A = \sum_\lambda \lambda P_\lambda$	?
① $p(\lambda) = tr(\rho P_\lambda) \geq 0 \Leftrightarrow P_\lambda \geq 0$	$E_\alpha \geq 0$, $p_\alpha = tr(\rho E_\alpha) \geq 0$ ↑ POVM element
② Normalization: $\sum_\lambda p(\lambda) = 1 \Leftrightarrow \sum_\lambda P_\lambda = 1$	$\sum_\alpha E_\alpha = 1$, $\sum_\alpha p_\alpha = 1$ ↑ resolution of identity / completeness
③ For each λ there are states that give certainty (support restricted to S_λ) $\Leftrightarrow P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_{\lambda'}$	Generally there is no state that gives certainty for a result α
④ Von-Neumann measurement (post measurement state:) $\rho_\lambda = P_\lambda \rho P_\lambda / p(\lambda)$	?

Qubit examples

① $E_\uparrow = P_\uparrow$, $E_\downarrow = P_\downarrow$ One dimensional orthogonal projectors (ODOP)

② $E_1 = \frac{1}{2} |\bar{e}_z\rangle\langle\bar{e}_z| = \frac{1}{4}(\mathbb{1} + Z)$

$E_2 = \frac{1}{2} |-\bar{e}_z\rangle\langle-\bar{e}_z| = \frac{1}{4}(\mathbb{1} - Z)$

$E_3 = \frac{1}{2} |\bar{e}_x\rangle\langle\bar{e}_x| = \frac{1}{4}(\mathbb{1} + X)$

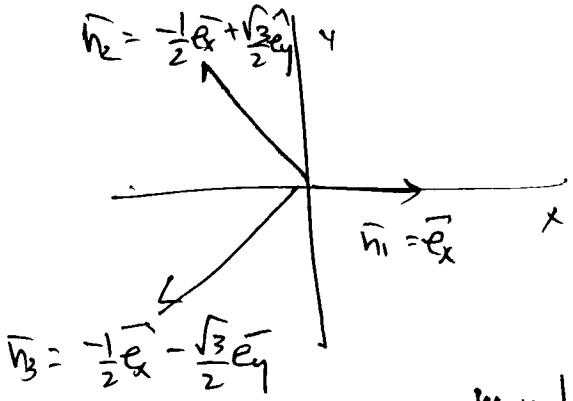
$E_4 = \frac{1}{2} |-\bar{e}_x\rangle\langle-\bar{e}_x| = \frac{1}{4}(\mathbb{1} - X)$

Coin flip to decide on measurement of Z-spin or X-spin

$\left(\text{max prob for any result}\right) = 1/2$

Swabshi

③ Trine measurement



$$E_1 = \frac{2}{3}|\bar{n}_1\rangle\langle\bar{n}_1| = \frac{1}{3}(\mathbb{1} + X)$$

$$E_2 = \frac{2}{3}|\bar{n}_2\rangle\langle\bar{n}_2| = \frac{1}{3}\left(\mathbb{1} - \frac{X}{2} + \frac{\sqrt{3}}{2}Y\right)$$

$$E_3 = \frac{2}{3}|\bar{n}_3\rangle\langle\bar{n}_3| = \frac{1}{3}\left(\mathbb{1} - \frac{X}{2} - \frac{\sqrt{3}}{2}Y\right)$$

max probability for any result = $2/3$

④ $$E_1 = \frac{1}{3}(\mathbb{1} + X) = \frac{2}{3}|\bar{e}_x\rangle\langle\bar{e}_x|$$

$$E_{23} = \underbrace{E_2 + E_3}_{\text{Coarse graining}} = \frac{2}{3}\left(\mathbb{1} - \frac{1}{2}X\right) = \frac{1}{3}|\bar{e}_x\rangle\langle\bar{e}_x| + |-\bar{e}_x\rangle\langle-\bar{e}_x|$$

⑤ General $$E_\alpha = q_\alpha(\mathbb{1} + \bar{r}_\alpha \cdot \bar{\sigma})$$

$$E_\alpha \geq 0 \Rightarrow q_\alpha \geq 0$$
$$|r_\alpha| \leq 1$$

$$E_\alpha \leq \mathbb{1} \Rightarrow q_\alpha(1 + |\bar{r}_\alpha|) \leq 1$$

$$\sum_\alpha E_\alpha = \mathbb{1} \Rightarrow \sum_\alpha q_\alpha = 1$$

$$\sum_\alpha q_\alpha \bar{r}_\alpha = 0$$

Harmonic oscillator example:

Simultaneous measurement of x & p

$$E_\alpha = \frac{1}{\pi}|\alpha\rangle\langle\alpha|$$

$$\int d^2\alpha\, E_\alpha = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{1}$$

$$Q(\alpha) = \mathrm{tr}(\rho E_\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

↳ Q-function
(Husimi distribution)

Swabeshi