PHY 4105: Quantum Information Theory
Tutorial Problems

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(Dated: September 27, 2013)

1. Show that an operator ρ is a pure state density matrix if and only if $\rho = \rho^\dagger$, $\mathrm{tr}(\rho^2) = 1$ and $\mathrm{tr}(\rho^3) = 1$.

2. Consider a state which has bloch vector $\vec{s} = \vec{e_z}/2$,

$$\rho = \frac{3}{4}|e_z\rangle\langle e_z| + \frac{1}{4}|-e_z\rangle\langle -e_z| = \frac{1}{2}\left(\mathbb{1} + \frac{1}{2}\sigma_z\right),$$

   (a) Give ensemble decompositions for ρ in which all pure states in the decomposition are selected from those with $\langle\sigma_z\rangle = 1/2$, (i) Two such states, (ii) Three such states, (iii) All such states.

   (b) Give an ensemble decomposition for ρ that includes all pure states (as an integral over the bloch sphere).

3. Consider the Greenberger-Horn-Zeilinger (GHZ) state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle - |111\rangle\big).$$

   (a) Show that the GHZ state is the $+1$ eigenstate of the operators $X \otimes Y \otimes Y$, $Y \otimes X \otimes Y$ and $Y \otimes Y \otimes X$.

   (b) Use the result above to argue that each qubit has well defined values for X and Y. For qubit j denote these values by x_j and y_j. What does local realism predict for the product of outcomes of measurements of X on each qubit.

   (c) What does quantum mechanics predict for the product of outcomes of measurements of X on each qubit.

4. Consider two systems, A of dimension d_A and B of dimension d_B. For an arbitrary pure state $|\Psi\rangle = \sum_{j,k} c_{jk}|e_j,\, f_k\rangle$,

   (a) Show that $|\Psi\rangle$ can be brought to the Schmidt form by using the singular value decomposition of the matrix whose entries are c_{jk} and find the Schmidt vectors for the two sub-systems in terms of the unitary matrices involved in the Schmidt decomposition.

   (b) Now suppose the two systems have the same dimension d. A maximally entangled state of A and B is one such that the marginal density operators are maximally mixed, i.e., $\rho_A = \mathbb{1}_A/d$ and $\rho_B = \mathbb{1}_B/d$. Find the conditions on c_{jk} such that $|\psi\rangle$ is maximally entangled, and discuss what this means for the singular values of c_{jk} and thus for the Schmidt coefficients.