PHY 4105: Quantum Information Theory
Lecture 11

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Multiple quantum systems

Consider two quantum systems. System A has a d_A dimensional Hilbert space while system B has a d_B dimensional Hilbert space. What is the dimension and nature of the state space of the composite system AB? In the composite state space there must be elements corresponding to the situation where A is in a pure state $|\psi\rangle$ and B is in another pure state $|\phi\rangle$. We denote the composite state as

$$|\psi\rangle \otimes |\phi\rangle,$$

and call such a state a product state. The symbol $\otimes$ (read as "o-times") for the time being is just a separator that distinguishes the state of A from the state of B. In general we can take arbitrary vectors from the Hilbert space $\mathcal{H}_A$ of A and the Hilbert space $\mathcal{H}_B$ of B, irrespective of whether they are normalized or not and construct composite states as above. The set of all such ordered pairs of one vector from $\mathcal{H}_A$ and another from $\mathcal{H}_B$ (set of all product states) is called the *cartesian product* of $\mathcal{H}_A$ and $\mathcal{H}_B$. This set does not form a vector space.

If $|\psi\rangle$ is itself a superposition, $|\psi\rangle = a|\chi\rangle + b|\xi\rangle$, then we can certainly write

$$|\psi\rangle \otimes |\phi\rangle = (a|\chi\rangle + b|\xi\rangle) \otimes |\phi\rangle = a|\chi\rangle \otimes |\phi\rangle + b|\xi\rangle \otimes |\phi\rangle.$$

The last equality means that when we say A is a superposition while B is in the state $|\phi\rangle$ then that is also equivalent to superposing the two composite states $|\chi\rangle \otimes |\phi\rangle$ and $|\xi\rangle \otimes |\phi\rangle$. We can say something similar when B is a superposition also but this rather innocuous statement has rather far reaching consequences including mandating the phenomena of quantum entanglement.

As we mentioned earlier, it is easy to see that the cartesian product space is not a vector space because arbitrary superpositions of the kind

$$a|\psi\rangle \otimes |\phi\rangle + b|\chi\rangle \otimes |\zeta\rangle,$$

are not product vectors. We have seen before that quantum systems are in general described by vectors in complex vector spaces. This is the mathematical statement of the principle of superposition. In accordance with this principle, the appropriate state space for composite systems is not the cartesian product space of pure states of the component systems but rather it is the entire complex vector space spanned by the cartesian product states. Once we make this assignment then we see that composite (pure) states like the one above which are not product states are allowed ones and precisely these states are the entangled ones.

The state space of the composite system is the *tensor product* of $\mathcal{H}_A$ and $\mathcal{H}_B$, denoted as

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

The symbol $\otimes$ is from now on called the tensor product. The inner product on the tensor product space is defined by defining the inner product of two product states as

$$(\langle\psi| \otimes \langle\phi|)(|\chi\rangle \otimes |\zeta\rangle) = \langle\psi|\chi\rangle\langle\phi|\zeta\rangle,$$

and extending this definition to all vectors in $\mathcal{H}_{AB}$ using the complex bi-linearity of the inner product.

Any vector $|\Psi\rangle$ in $\mathcal{H}_{AB}$ can be written as superposition of product states. In particular all the vectors in the product can be expanded in terms of basis vectors $|e_j\rangle$ for $\mathcal{H}_A$ and $|f_k\rangle$ for $\mathcal{H}_B$, and the bi-linearity of the tensor product can be used to write $|\Psi\rangle$ as a linear combination of orthonormal product vectors $|e_j\rangle \otimes |f_k\rangle$ showing that these vectors form a basis for $\mathcal{H}_{AB}$. The number of such vectors is $d_A d_B$ showing that the dimension of the tensor product space is the product of the dimensions of the component spaces. We often do not write the tensor product symbol explicitly and may denote the basis vectors of $\mathcal{H}_{AB}$ using any one of the following forms,

$$|e_j\rangle \otimes |f_k\rangle = |e_j\rangle|f_k\rangle = |e_j, f_k\rangle = |j, k\rangle = |jk\rangle.$$

The expansion of an arbitrary vector in $\mathcal{H}_{AB}$ looks like

$$|\Psi\rangle = \sum_{jk} |e_j, f_k\rangle\langle e_j f_k|\Psi\rangle = \sum_{jk} c_{jk}|e_j, f_k\rangle.$$

As we had done previously for states of single systems, the expansion coefficients $c_{jk} = \langle e_j, f_k|\Psi\rangle$ can be written as a column vector (matrix representation in a particular basis):

$$|\Psi\rangle = \begin{pmatrix} c_{11} \\ \vdots \\ c_{1d_B} \\ c_{21} \\ \vdots \\ c_{2d_B} \\ \vdots \\ c_{d_A 1} \\ \vdots \\ c_{d_A d_B} \end{pmatrix} = \begin{pmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_{d_A} \end{pmatrix}.$$

In the second form we have group together the coefficients into d_B dimensional vectors of the form

$$\vec{c}_j = \begin{pmatrix} c_{j1} \\ \vdots \\ c_{jd_B} \end{pmatrix},$$

which is also equivalent to writing

$$|\Psi\rangle = \sum_j |e_j\rangle \otimes \Big(\sum_k c_{jk}|f_k\rangle\Big).$$

For a product vector the expansion coefficients c_{jk} are the outer product of the expansion coefficients for $|\psi\rangle = \sum_j a_j|e_j\rangle$ and $|\phi\rangle = \sum_j b_k|f_k\rangle$:

$$c_{jk} = \langle e_j, f_k|(|\psi\rangle \otimes |\phi\rangle) = \langle e_j|\psi\rangle\langle f_k||\phi\rangle = a_j b_k, \qquad \vec{c}_j = a_j \begin{pmatrix} b_1 \\ \vdots \\ b_{d_B} \end{pmatrix}.$$

We can now think of a "partial inner product" $\langle \phi_B|\Psi_{AB}\rangle$. It is that ket in $\mathcal{H}_A$ whose inner product with any vector $|\psi_A\rangle$ in $\mathcal{H}_A$ is the same as the inner product of $\langle\psi_A| \otimes \langle\phi_B|$ with $|\Psi_{AB}\rangle$, i.e.,

$$\langle\psi_A|(\langle\phi_B|\Psi_{AB}\rangle) = (\langle\psi_A| \otimes \langle\phi_B|)|\Psi_{AB}\rangle.$$

Explicitly in the product basis we have

$$\langle\phi_B|\Psi_{AB}\rangle = \sum_{jk} c_{jk}|e_j\rangle\langle\phi_B|f_k\rangle = \sum_j |e_j\rangle\Big(\sum_k b_k^* c_{jk}\Big).$$

For a product state, the partial inner product is

$$\langle\phi_B|\Psi_{AB}\rangle = \langle\phi_B|(|\psi_A\rangle \otimes |\xi_B\rangle) = |\psi_A\rangle\langle\phi_B|\xi_B\rangle.$$

We can define a partial inner product with respect to system A as well in an equivalent manner.

1. *Operators on tensor product space*

The basic operators on the tensor product space $\mathcal{H}_{AB}$ are outer products of the form

$$(|\psi\rangle \otimes |\phi\rangle)(\langle\chi| \otimes \langle\xi|) = |\psi\rangle\langle\chi| \otimes |\phi\rangle\langle\xi|.$$

The first form is the standard outer product notation in the tensor product space; we know how to handle this form since we have already defined the inner product of these kinds of vectors. The second form just a rearrangement of the vectors and that defines the tensor product of operators. We get a tensor product of operators on A and B by this rearrangement. This definition is extended to the product of any two bilinear operators by assuming that the tensor product is bilinear as well:

$$\begin{aligned} A \otimes B &= \Big(\sum_{jl} A_{jl}|e_j\rangle\langle e_l|\Big) \otimes \Big(\sum_{km} B_{km}|f_k\rangle\langle f_m|\Big) \\ &= \sum_{jklm} A_{jl}B_{km}|e_j\rangle\langle e_l| \otimes |f_k\rangle\langle_| f_m \\ &= \sum_{jklm} A_{jl}B_{km}|e_j, f_k\rangle\langle e_l, f_m|. \end{aligned}$$

In fact if we know that we are working in tensor product Hilbert space then we really should write all operators A acting on one of the Hilbert spaces, say $\mathcal{H}_A$ as

$$A \otimes \mathbb{1} = \sum_{jlk} A_{jl}|e_j\rangle\langle e_l| \otimes |f_k\rangle\langle f_k| = \sum_{jlk} A_{jl}|e_j, f_k\rangle\langle e_l, f_k|.$$

We often ignore this nicety unless the context does not make it clear what we are talking about. We also have products of tensor product operators as

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2.$$

An arbitrary operator O acting on tensor product space can be expanded as

$$O = \sum_{jklm} O_{jk,lm}|e_j, f_k\rangle\langle e_l, f_m|.$$

In matrix notation we get

$$O = \begin{pmatrix} O_{11,11} & \cdots & O_{11,1d_B} & \cdots & O_{11,d_A 1} & \cdots & O_{11,d_A d_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ O_{1d_B,11} & \cdots & O_{1d_B,1d_B} & \cdots & O_{1d_B,d_A 1} & \cdots & O_{1d_B,d_A d_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ O_{d_A 1,11} & \cdots & O_{d_A 1,1d_B} & \cdots & O_{d_A 1,d_A 1} & \cdots & O_{d_A 1,d_A d_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ O_{d_A d_B,11} & \cdots & O_{d_A d_B,1d_B} & \cdots & O_{d_A d_B,d_A 1} & \cdots & O_{d_A d_B,d_A d_B} \end{pmatrix} = \begin{pmatrix} O_{11} & \cdots & O_{1d_A} \\ \vdots & \ddots & \vdots \\ O_{d_A 1} & \cdots & O_{d_A d_A} \end{pmatrix},$$

where in the second form we have a matrix of matrices with each O_{jl} being a $(d_B \times d_B)$ matrix,

$$O_{jl} = \begin{pmatrix} O_{j1,l1} & \cdots & O_{j1,ld_B} \\ \vdots & \ddots & \vdots \\ O_{jd_B,l1} & \cdots & O_{jd_B,ld_B} \end{pmatrix}.$$

What we have done is to rewrite O as

$$O = \sum_{jl} |e_j\rangle\langle e_l| \otimes \Big(\sum_{km} O_{jk,lm}|f_k\rangle\langle_| f_m\Big).$$

Using the partial inner product of tensor product vectors we can define a partial matrix element of a composite operator O as

$$\langle\phi_B|O|\xi_B\rangle = \sum_{jl} |e_j\rangle\langle e_l|\Big(\sum_{km} O_{jk,lm}\langle\phi_B|f_k\rangle\langle f_m|\xi_B\rangle\Big),$$

which is an operator acting on system A. For a tensor product operator

$$\langle\phi_B|A \otimes B|\xi_B\rangle = A\langle\phi_B|B|\xi_B\rangle.$$

Similarly

$$\langle\phi_B|(A \otimes \mathbb{1})O|\xi_B\rangle = A\langle\phi_B|O|\xi_B\rangle, \quad \text{and} \quad \langle\phi_B|O(A \otimes \mathbb{1})|\xi_B\rangle = \langle\phi_B|O|\xi_B\rangle A.$$

In a similar manner we can always define a partial matrix element with respect to system A as $\langle\psi_A|O|\chi_A\rangle$.

We now have all the ingredients in place to talk about the partial trace operation.