PHY 4105: Quantum Information Theory
Lecture 14

Anil Shaji
School of Physics, IISER Thiruvananthapuram
(Dated: September 19, 2013)

Pauli representation of Bell basis states

We have seen that any operator can be expanded in the Pauli basis as

$$\rho = \frac{1}{4}\sum_{\alpha\beta} \rho_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta, \quad \text{with} \quad \rho_{\alpha\beta} = \text{tr}(\rho\sigma_\alpha \otimes \sigma_\beta), \quad \rho_{00} = 1.$$

We can do an expansion of this sort for the Bell basis states,

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|).$$

Using

$$\begin{aligned}
|00\rangle\langle 00| &= \frac{1}{2}(\mathbb{1} + Z) \otimes \frac{1}{2}(\mathbb{1} + Z) \\
|11\rangle\langle 11| &= \frac{1}{2}(\mathbb{1} - Z) \otimes \frac{1}{2}(\mathbb{1} - Z) \\
|00\rangle\langle 1| &= \frac{1}{2}(X + iY) \otimes \frac{1}{2}(X + iY) \\
|11\rangle\langle 00| &= \frac{1}{2}(X - iY) \otimes \frac{1}{2}(X - iY),
\end{aligned}$$

we get

$$|\Phi^+\rangle\langle\Phi^+| = |\beta_{00}\rangle\langle\beta_{00}| = \frac{1}{4}\big(\mathbb{1} \otimes \mathbb{1} + Z \otimes Z + X \otimes X - Y \otimes Y\big).$$

Similarly we have

$$\begin{aligned}
|\Phi^-\rangle\langle\Phi^-| &= |\beta_{10}\rangle\langle\beta_{10}| &= \frac{1}{4}\big(\mathbb{1} \otimes \mathbb{1} + Z \otimes Z - X \otimes X + Y \otimes Y\big) \\
|\Psi^+\rangle\langle\Psi^+| &= |\beta_{01}\rangle\langle\beta_{01}| &= \frac{1}{4}\big(\mathbb{1} \otimes \mathbb{1} - Z \otimes Z + X \otimes X + Y \otimes Y\big) \\
|\Psi^-\rangle\langle\Psi^-| &= |\beta_{11}\rangle\langle\beta_{11}| &= \frac{1}{4}\big(\mathbb{1} \otimes \mathbb{1} - Z \otimes Z - X \otimes X - Y \otimes Y\big)
\end{aligned}$$

The last state is rotationally invariant. This is because

$$u \otimes u(\sigma_j \otimes \sigma_j)u \otimes u = \sigma_k R_{kj} \otimes \sigma_l R_{lj} = \sigma_k \otimes \sigma_l (RR^T)_{kl} = \sigma_k \otimes \sigma_k.$$

It is rather useful to note that the Z operator represents a 180 degree rotation about the $z-$axis and X represents a 180 degree rotation around the $x-$axis and so on. In other words

$$ZXZ = -X, \quad ZYZ = -Y, \quad ZZZ = Z, \qquad XXX = X, \quad XYX = -Y, \quad XZX = -Z.$$

In general we can write the Pauli representation of the four Bell states as

$$|\beta_{ab}\rangle\langle\beta_{ab}| = \frac{1}{4}\big(\mathbb{1}\otimes\mathbb{1} + (-1)^b Z\otimes Z + (-1)^a X\otimes X - (-1)^{a+b} Y\otimes Y\big).$$

Inverting these relations we find that

$$\begin{aligned}
\mathbb{1}\otimes\mathbb{1} &= +|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{11}\rangle\langle\beta_{11}| \\
Z\otimes Z &= +|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{01}\rangle\langle\beta_{01}| - |\beta_{10}\rangle\langle\beta_{10}| - |\beta_{11}\rangle\langle\beta_{11}| \\
X\otimes X &= +|\beta_{00}\rangle\langle\beta_{00}| - |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{10}\rangle\langle\beta_{10}| - |\beta_{11}\rangle\langle\beta_{11}| \\
Y\otimes Y &= -|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{10}\rangle\langle\beta_{10}| - |\beta_{11}\rangle\langle\beta_{11}|.
\end{aligned}$$

The four operators above commute with each other but any two will form a complete commuting set with the other two being products of the two. We also know that

$$Z\otimes Z|\beta_{ab}\rangle = (-1)^b|\beta_{ab}\rangle,$$

so that a measurement of $Z\otimes Z$ determines the parity bit. Similarly a measurement of $X\otimes X$ determines the phase bit:

$$X\otimes X|\beta_{ab}\rangle = (-1)^a|\beta_{ab}\rangle.$$

We also have

$$Y\otimes Y|\beta_{ab}\rangle = -ZX\otimes ZX|\beta_{ab}\rangle = -(-1)^{a+b}|\beta_{ab}\rangle.$$

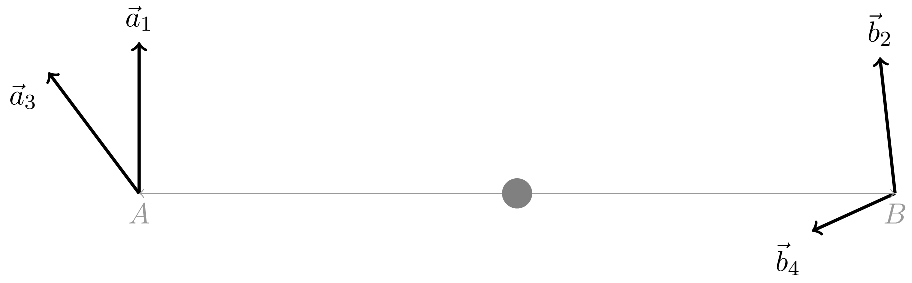
CHSH Bell inequality (Clauser-Horne-Shimony-Holt-Bell)

The question addressed by the CHSH inequality (and Bell's inequality) is whether quantum mechanics is a local realistic theory. So what does local realism mean? It is closely related to the question we have asked before as to whether we can attribute specific states to subsystems of a composite quantum system. We have seen that indeed we can assign mixed states to the subsystems when the overall state is entangled. However is that consistent with the rules of classical physics or is it different?

Suppose we are going to measure the operator $\sigma_{\vec{a}}$ on a qubit. Prior to the measurement can be ascribe a value to the projection of the state along $\vec{a}$ as an objective property of the system A. Similarly for subsystem B of the composite system AB can we assign the value $\sigma_{\vec{b}} = \pm 1$ as an objective property of system B? Realism implies that we can indeed assign such values as objective properties of the two subsystems.

Another requirement of classical physics (including relativity) is that if the two subsystems happen to be well segregated (or maybe even well separated in space), the a measurement of subsystem A should not affect B and in particular it should not affect the objective values of B. This no-disturbance is enforced by locality. So together you have local realism.

Consider the following experimental set up. A pair of qubits labeled A and B is produced by some physical process like, say, the decay of a spin zero particle into two spin 1/2 particles and then they move away in opposite directions. When they are well separated, Stern-Gerlach (projective) type measurements are done on both particles. The experimenter at either end has two choices for the direction along which the Stern-Gerlach apparatus can be oriented. These directions are labeled by $\vec{a}_1$ and $\vec{a}_3$ on subsystem A and $\vec{b}_2$ and $\vec{b}_4$ on subsystem B as shown in the figure below:



Now consider the quantity

$$\left|\sigma_{\vec{a}_1}(\sigma_{\vec{b}_2} - \sigma_{\vec{b}_4}) + \sigma_{\vec{a}_3}(\sigma_{\vec{b}_2} + \sigma_{\vec{b}_4})\right| \equiv |\hat{S}|,$$

where $\sigma_{\vec{a}}$ and $\sigma_{\vec{b}}$ denote the outcome of the measurement on the respective qubit. If we assume local realism then each of these measurement results have a value ± 1 independent of the measurement setting used in the particular instance of the experiment by each experimenter. This means that we have the following possible values for the measurement result as well as for $|\hat{S}|$ assigned as objective properties of subsystems A and B:

$\sigma_{\vec{a}_1}$	$\sigma_{\vec{a}_3}$	$\sigma_{\vec{b}_2}$	$\sigma_{\vec{b}_4}$	$\lvert\hat{S}\rvert$
1	1	1	1	2
1	1	1	-1	2
1	1	-1	1	2
1	1	-1	-1	2
1	-1	1	1	2
1	-1	1	-1	2
1	-1	-1	1	2
1	-1	-1	-1	2
-1	1	1	1	2
-1	1	1	-1	2
-1	1	-1	1	2
-1	1	-1	-1	2
-1	-1	1	1	2
-1	-1	1	-1	2
-1	-1	-1	1	2
-1	-1	-1	-1	2

So we see that assuming local realism we get

$$\left|\sigma_{\vec{a}_1}(\sigma_{\vec{b}_2} - \sigma_{\vec{b}_4}) + \sigma_{\vec{a}_3}(\sigma_{\vec{b}_2} + \sigma_{\vec{b}_4})\right| \equiv |\hat{S}| = 2$$

Now suppose we have an ensemble of identically prepared qubit pairs and we make the measurements above and take the average the function given above of measurement results then using the fact that

$$|\langle\cdots\rangle| \leq \langle|\cdots|\rangle,$$

we get

$$\left|C(\vec{a}_1,\vec{b}_2)+C(\vec{a}_3,\vec{b}_2)+C(\vec{a}_3,\vec{b}_4)-C(\vec{a}_1,\vec{b}_4)\right| \leq 2, \qquad C(\vec{a}_i,\vec{b}_j) \equiv \langle\sigma_{\vec{a}_i}\sigma_{\vec{b}_j}\rangle.$$

This is the CHSH inequality. Any local realistic system has to satisfy this inequality.

Now let us look whether indeed a pair of quantum systems in an arbitrary bipartite state does satisfy the CHSH inequality. Let us assume that the pair of qubits is in the singlet state,

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

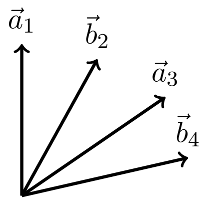
For the singlet state we have

$$C(\vec{a},\vec{b}) = \langle\sigma_j a_j \otimes \sigma_k b_k\rangle = a_j b_k\langle\sigma_j\otimes\sigma_k\rangle = a_j b_k(-\delta_{jk}) = -\vec{a}\cdot\vec{b},$$

using

$$|\beta_{11}\rangle\langle\beta_{11}| = \frac{1}{4}\big(\mathbb{1}\otimes\mathbb{1} - Z\otimes Z - X\otimes X - Y\otimes Y\big).$$

Let us choose the vectors $\vec{a}_1$, $\vec{a}_3$, $\vec{b}_2$ and $\vec{b}_4$ as shown below,



The angle between each adjacent pair of vectors is fixed at θ. So

$$C(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b} = -\cos\theta_{ab},$$

and

$$S = \left|C(\vec{a}_1,\vec{b}_2)+C(\vec{a}_3,\vec{b}_2)+C(\vec{a}_3,\vec{b}_4)-C(\vec{a}_1,\vec{b}_4)\right| = \left|3\cos\theta - \cos 3\theta\right|,$$

which for $\theta=\pi/4$ has the value $2\sqrt{2}>2$. A different way of seeing this is that for $\theta=\pi/4$ we can choose

$$\vec{a}_1 = \hat{e}_x, \quad \vec{b}_2 = \frac{1}{\sqrt{2}}(\hat{e}_y+\hat{e}_x), \quad \vec{a}_3 = \hat{e}_y, \quad \vec{b}_4 = \frac{1}{\sqrt{2}}(\hat{e}_y-\hat{e}_x),$$

so that

$$\begin{aligned} S &= \left\langle X\otimes\frac{1}{\sqrt{2}}(Y+X) + Y\otimes\frac{1}{\sqrt{2}}(Y+X) + Y\otimes\frac{1}{\sqrt{2}}(Y-X) - X\otimes\frac{1}{\sqrt{2}}(Y-X)\right\rangle \\ &= \sqrt{2}\langle X\otimes X + Y\otimes Y\rangle \\ &= 2\sqrt{2}\langle|\beta_{01}\rangle\langle\beta_{01}| - |\beta_{11}\rangle\langle\beta_{11}|\rangle. \end{aligned}$$

We see that S is maximum if the state in which the expectation value is taken is either $|\beta_{01}\rangle$ (correlations) or the state $|\beta_{11}\rangle$ (anticorrelations).