PHY 4105: Quantum Information Theory
Lecture 15

Anil Shaji
School of Physics, IISER Thiruvananthapuram
(Dated: September 24, 2013)

Quantum teleportation

Alice and Bob each has one of the pair of qubits in the entangled state

$$|\beta_{00}^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Victor hands over to Alice a qubit in the state $|\psi^V\rangle$ so that the total three qubit state is at this time,

$$|\Psi\rangle = |\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle = \sum_{a,b} |\beta_{ab}^{VA}\rangle\langle\beta_{ab}^{VA}|\Psi\rangle,$$

where $\langle\beta_{ab}^{VA}|\Psi\rangle$ is a vector in Bob's Hilbert space times the magnitude of the overlap of $|\rangle\beta_{ab}^{VA}$ with the part of $|\Psi\rangle$ in the VA Hilbert space. Essentially we have expended the VA part of the state in the bell basis for those two qubits. Now consider the relative state

$$\begin{aligned}
\langle\beta_{00}^{VA}|\Psi\rangle &= \langle\beta_{00}^{VA}|\big(|\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle\big) \\
&= \frac{1}{2}\big(\langle 00| + \langle 11|\big)\big(|\psi^V\rangle \otimes (|00\rangle + |11\rangle)\big) \\
&= \frac{1}{2}\big(\langle 0|\psi^V\rangle|0\rangle_B + \langle 1|\psi^V\rangle|1\rangle_B\big) \\
&= \frac{1}{2}|\psi^B\rangle.
\end{aligned}$$

Using

$$|\beta_{ab}\rangle = Z^a X^b \otimes \mathbb{1}|\beta_{00}\rangle \qquad \Leftrightarrow \qquad \langle\beta_{ab}|X^b Z^a \otimes \mathbb{1},$$

we get

$$\begin{aligned}
\langle\beta_{ab}^{VA}|\Psi\rangle &= \langle\beta_{00}^{VA}|\big(X^b Z^a \otimes \mathbb{1}\big)\big(|\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle\big) \\
&= \frac{1}{2}\big(\langle 00| + \langle 11|\big)\big(X^b Z^a|\psi^V\rangle \otimes (|00\rangle + |11\rangle)\big) \\
&= \frac{1}{2}\big(\langle 0|X^b Z^a|\psi^V\rangle|0\rangle_B + \langle 1|X^b Z^a|\psi^V\rangle|1\rangle_B\big) \\
&= \frac{1}{2}X^b Z^a|\psi^B\rangle.
\end{aligned}$$

So we have

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{2}\sum_{ab} |\beta_{ab}^{VA}\rangle \otimes X^b Z^a|\psi^B\rangle \\
&= \frac{1}{2}\big(|\beta_{00}^{VA}\rangle \otimes |\psi^B\rangle + |\beta_{10}^{VA}\rangle \otimes Z|\psi^B\rangle + |\beta_{01}^{VA}\rangle \otimes X|\psi^B\rangle + |\beta_{11}^{VA}\rangle \otimes XZ|\psi^B\rangle\big).
\end{aligned}$$

To teleport the qubit given to her by Victor, Alice takes the pair of states in her hand and measures it in the bell basis. She would get one of four results with equal chance. After the measurement there is no trace of $|\psi^V\rangle$ left at Alices' end and all that is left is a combined bell state of the VA qubits. She then sends over the results of her measurements in the form of two bits, a and b over to Bob over a classical communication channel. Bob can drive his qubit to the state $|\psi\rangle$ by applying $X^b Z^a$ to his qubit. Thus the arbitrary qubit state $|\psi\rangle$ gets teleported from Alice to Bob.

One has to see teleportations in the context of indistinguishability of quantum particles. The particle is essentially labelled only by its state except for labels common to all the qubits. For instance if the qubits are spin-1/2 electrons, then one electron is indistinguishable from the other except that each of the three are labeled by the specific states they are in. So when the state of the qubit handed to Alice by Victor is transferred over to the electron with Bob, then effectively Bob is indeed getting Victors qubit in all respects.

Polar decomposition and singular values

Given any operator (not necessarily Hermitian), $A^\dagger A$ and $AA^\dagger$ are positive operators with the same eigenvalues whose square roots are the *singular values* of A. Moreover there exists a unitary operator U such that

$$A = U\sqrt{A^\dagger A} = \sqrt{AA^\dagger}U,$$

where the above decomposition of A is called the Polar decomposition. The unitary U is unique if $A^\dagger A$ is invertible (no null space). For the two positive operators we have the eigen-decomposition,

$$\begin{aligned} A^\dagger A &= \sum_j \lambda_j^2 |e_j\rangle\langle e_j| \\ AA^\dagger &= \sum_j \lambda_j^2 |f_j\rangle\langle f_j|. \end{aligned}$$

The vectors $|e_j\rangle$ are called the right singular vectors of A and $|f_j\rangle$ are called the left singular vectors of A. In terms of the left and right singular vectors we can expand the operator as

$$A = \sum_j \lambda_j |f_j\rangle\langle e_j|.$$

Note that this reduces to the eigenvalue decomposition when the left and right singular eigenvectors are duals of each other. In that case $A^\dagger A = AA^\dagger$ which means that A is a normal operator as expected. We can also identify the unitary U above as

$$U = \sum_j |f_j\rangle\langle e_j| \qquad U|e_j\rangle = |f_j\rangle,$$

so that

$$A = U\sqrt{A^\dagger A} = \Big(\sum_j |f_j\rangle\langle e_j|\Big) \sum_k \lambda_j |e_k\rangle\langle e_k| = \sum_j \lambda_j |f_j\rangle\langle e_j| = \sqrt{AA^\dagger}U.$$

The freedom in U is with respect to how it maps vectors in the null space of $A^\dagger A$ to vectors in the null space of $AA^\dagger$. If there are no zero eigenvalues for $A^\dagger A$ ($A^\dagger A$ is invertible) then this freedom is not present and U is unique.

Related to the Polar decomposition of an operator we also have an singular value decomposition as

$$A = V\Lambda W^\dagger, \qquad \Lambda = \sum_j |g_j\rangle\langle g_j|,$$

where $|g_j\rangle$ is an arbitrary orthonormal basis in which Λ is diagonal. The eigenvalues of Λ are the singular values of A. We can easily see

$$A = \sum_j \lambda_j V|g_j\rangle\langle g_j|W^\dagger, \qquad V|g_j\rangle = |f_j\rangle \quad \text{and} \quad \langle g_j|W^\dagger = \langle e_j|.$$

Relative state decomposition

Consider a bipartite pure state,

$$|\Psi\rangle = \sum_\alpha |f_\alpha\rangle\langle f_\alpha|\Psi\rangle = \sum_\alpha \sqrt{p_\alpha}|\phi_\alpha\rangle \otimes |f_\alpha\rangle, \qquad \langle f_\alpha|\Psi\rangle \equiv \sqrt{p_\alpha}|\phi_\alpha\rangle.$$

We have found the relative state with respect to the orthogonal basis $|f_\alpha\rangle$ on B of the bipartite state $|\Psi\rangle$. We then have

$$\begin{aligned} \rho_A &= \mathrm{tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{tr}_B\Big(\sum_{\alpha,\beta} \sqrt{p_\alpha}\sqrt{p_\beta}|\phi_\alpha\rangle\langle\phi_\beta| \otimes |f_\alpha\rangle\langle f_\beta|\Big) \\ &= \sum_{\alpha,\beta} \sqrt{p_\alpha}\sqrt{p_\beta}|\phi_\alpha\rangle\langle\phi_\beta|\langle f_\beta|f_\alpha\rangle \\ &= \sum_\alpha p_\alpha|\phi_\alpha\rangle\langle\phi_\alpha|. \end{aligned}$$

The relative state decomposition of the bipartite pure state directly gives us an ensemble decomposition for one of the subsystems.

Schmidt decomposition

Any bipartite pure state can be written as

$$|\Psi\rangle = \sum_j \sqrt{\lambda_j}|e_j\rangle \otimes |f_j\rangle,$$

where $|e_j\rangle$ and $|f_j\rangle$ form orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$. The $\sqrt{\lambda_j}$ are non-negative *Schmidt* coefficients and as consequence of the decomposition we have

$$\rho_A = \sum_j \lambda_j|e_j\rangle\langle e_j|, \qquad \rho_B = \sum_j |f_j\rangle\langle f_j|.$$

The Schmidt coefficients provide a complete characterization of the entanglement in bipartite pure states. The coefficients are invariant under local unitaries. There is no Schmidt like decomposition for three or more systems.

Proof: Diagonalize

$$\rho_B = \mathrm{tr}_A(|\Psi\rangle\langle\Psi|) = \sum_j \lambda_j |f_j\rangle\langle f_j|.$$

Now form the relative state decomposition of $|\Psi\rangle$ with respect to the eigenvectors $|f_j\rangle$ of ρ_B as

$$|\Psi\rangle = \sum_j |\phi_j\rangle \otimes |f_j\rangle, \qquad |\phi_j\rangle \equiv \langle f_j|\Psi\rangle.$$

Now we have

$$\begin{aligned} \rho_B &= \mathrm{tr}_A(|\Psi\rangle\langle\Psi|) = \mathrm{tr}_A\Big(\sum_{j,k} |\phi_j\rangle\langle\phi_k| \otimes |f_j\rangle\langle f_k|\Big) \\ &= \sum_{j,k} \langle\phi_j|\phi_k\rangle |f_j\rangle\langle f_k| \\ &= \sum_j \lambda_j |f_j\rangle\langle f_j|. \end{aligned}$$

Comparing the last two equations we have

$$\langle\phi_j|\phi_k\rangle = \lambda_j \delta_{jk}.$$

This means that $|\phi_j\rangle$ form an orthogonal set. Defining the orthonormal set

$$|e_j\rangle = \frac{|\phi_j\rangle}{\sqrt{\lambda_j}},$$

we get

$$|\Psi\rangle = \sum_j \sqrt{\lambda_j} |e_j\rangle \otimes |f_j\rangle.$$

Purification

A pure state $|\Psi\rangle$ is a purification of ρ_A if $\rho_A = \mathrm{tr}(|\Psi\rangle\langle\Psi|)$. A purification is a useful analytical tool, replacing a mixed state with a pure-state analysis in a larger space (The church of the larger Hilbert space).

- Ensemble decomposition: $\rho_A = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$
- Purification: $\Psi = \sum_\alpha \sqrt{p_\alpha} |\psi_\alpha\rangle \otimes |f_\alpha\rangle$.

The purification is a relative state decomposition with respect to a set of orthonormal vectors $|f_\alpha\rangle$ in $\mathcal{H}_B$.