PHY 4105: Quantum Information Theory
Lecture 17

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Measurement models

Let us now see how we can describe the situation where we model a quantum measurement as a process in which the quantum system is coupled to a measuring device and then the measurement is completed by observing the apparatus and inferring some property of the quantum system.

We have a system Q in the state ρ which is brought in contact with an *ancilla* system A (which can be the measuring device) in a state

$$\sigma = \sum_k \lambda_k |e_k\rangle\langle e_k|,$$

where the states $|e_k\rangle$ are eigenstates of σ. The two systems interact for some time with the interaction described by a unitary operator U acting on the joint system of Q and A. The ancilla is then subject to a vonNeumann measurement described by the orthogonal projectors

$$P_\alpha = \sum_j |f_{\alpha j}\rangle\langle f_{\alpha j}|,$$

The α index denotes the state of the system that corresponds to the subspace indexed by j of the ancilla. In other words there might be more than one state of the ancilla corresponding to a given state of the system. Typically this is the case since the ancilla is taken to be a larger system. The states $|f_{\alpha j}\rangle$ satisfies the completion relation,

$$\mathbb{1}_A = \sum_{\alpha,j} |f_{\alpha j}\rangle\langle f_{\alpha j}| = \sum_\alpha P_\alpha.$$

The Greek index α therefore labels the outcome of the measurement. If the ancilla is observed in the subspace S_α after the measurement, then we can compute the un-normalized state of the system after the measurement using the standard rules of vonNeumann measurements on the QA system and then tracing out the ancilla as

$$\mathrm{tr}_A(P_\alpha U \rho \otimes \sigma U^\dagger P_\alpha) = \mathrm{tr}_A(P_\alpha U \rho \otimes \sigma U^\dagger) = \mathcal{A}_\alpha \rho,$$

where $\mathcal{A}_\alpha$ is a linear map on system density operators. Any linear map on density operators is called a super-operator and the super-operator defined above is called a quantum operation. This particular method of defining a set of quantum operations in terms of the interaction of the system with an ancilla followed by a measurement of the ancilla is called a *measurement model.*

Inserting the expression we have for the ancilla projector P_α and the initial state of the ancilla into the expression above gives us a form for the quantum operations only in terms of system operators as

$$\mathcal{A}_\alpha(\rho) = \sum_{j,k} \sqrt{\lambda_k}\langle f_{\alpha j}|U|e_k\rangle\rho\langle e_k|U^\dagger|f_{\alpha j}\rangle\sqrt{\lambda_k} = \sum_{j,k} A_{\alpha jk}\rho A^\dagger_{\alpha jk}.$$

The operators

$$A_{\alpha jk} = \sqrt{\lambda_k}\langle f_{\alpha j}|U|e_k\rangle,$$

are said to provide an operator sum decomposition (Kraus decomposition) of the quantum operation $\mathcal{A}_\alpha$ and are therefore called operation elements or Kraus operators. The Kraus operators are defined by the relative states in the decomposition of $U|\psi\rangle \otimes |e_k\rangle$ relative to the ancilla basis $|f_{\alpha j}\rangle$:

$$U|\psi\rangle \otimes |e_k\rangle = \sum_{\alpha,j}\langle f_{\alpha j}|U|e_k\rangle|\psi\rangle \otimes |f_{\alpha j}\rangle = \sum_{\alpha,j} \frac{1}{\sqrt{\lambda_k}} A_{\alpha jk}|\psi\rangle \otimes |f_{\alpha j}\rangle.$$

The Kraus operators satisfy the completeness relation,

$$\sum_{\alpha,j,k} A^\dagger_{\alpha jk}A_{\alpha jk} = \sum_{\alpha jk} \lambda_k\langle e_k|U^\dagger|f_{\alpha j}\rangle\langle f_{\alpha j}|U|e_k\rangle = \mathrm{tr}_A(U^\dagger U\sigma) = \mathrm{tr}_A(\mathbb{1} \otimes \mathbb{1}\sigma) = \mathbb{1}.$$

The probability of getting the result α in the measurement of the ancilla is, from the standard rules for an orthogonal projective measurement, is

$$p_\alpha = \mathrm{tr}(P_\alpha U\rho \otimes U^\dagger) = \mathrm{tr}(\mathcal{A}_\alpha(\rho)) = \mathrm{tr}\left(\rho \sum_{jk} A^\dagger_{\alpha jk}A_{\alpha jk}\right) = \mathrm{tr}(\rho E_\alpha).$$

The operators

$$E_\alpha = \sum_{jk} A^\dagger_{\alpha jk}A_{\alpha jk} = \sum_{jk} \lambda_k\langle e_k|U^\dagger|f_{\alpha j}\rangle\langle f_{\alpha j}|U|e_k\rangle = \mathrm{tr}_A(U^\dagger P_\alpha U P_\alpha),$$

are positive operators and satisfy the completeness relation as shown above. Any measurement model therefore gives rise to a POVM that describes the measurement statistics. The normalized post measurement state conditioned on the result α is

$$\rho_\alpha = \frac{\mathcal{A}_\alpha(\rho)}{tr\Big(\mathcal{A}_\alpha(\rho)\Big)} = \frac{\mathcal{A}_\alpha(\rho)}{p_\alpha} = \frac{1}{p_\alpha}\sum_{j,k} A_{\alpha jk}\rho A^\dagger_{\alpha jk}.$$

If we are ignorant about the result of the measurement on the ancilla then the post-measurement state is obtained by averaging over the possible measurement results,

$$\rho' = \sum_\alpha p_\alpha\rho_\alpha = \sum_\alpha \mathcal{A}_\alpha(\rho) = \mathrm{tr}_A(U\rho \otimes \sigma U^\dagger) = \sum_{\alpha,j,k} A_{\alpha jk}\rho A^\dagger_{\alpha jk} \equiv \mathcal{A}(\rho).$$

This is in fact the dynamics one would get in the absence of any measurement on the ancilla or, formally, for a completely uninformative measurement on the ancilla ($P_\alpha = \mathbb{1}_A$).

A primary quantum system that is exposed to an initially uncorrelated environment always has dynamics described by a quantum operation, whether or not a measurement is made on the environment. If we do make a measurement on the environment, the system state after the dynamics is conditioned on the result of the measurement through the projection operator P_α. The corresponding quantum operation $\mathcal{A}$ is said to be trace decreasing because the trace of the output is generally smaller than the trace of the input, the reduction factor being the probability for the measurement result α. If we do not make a measurement on the environment, we have an open-system dynamics described by the operation $\mathcal{A}$, which is said to be trace preserving because the trace of the output is the same as the trace of the input. Formally $\mathcal{A}$ is the quantum operation for a completely uninformative measurement on the ancilla, which has one outcome $P_\alpha = \mathbb{1}_A$. Moreover, we can think of any trace-preserving open-system dynamics as coming from an environment that "monitors" the system, even though we acquire none of the monitored information; this monitoring destroys quantum coherence in Q, a process called decoherence.

The indices j and k that are summed over in the operator sum representation of $\mathcal{A}$ represents information that we could have had but do not have. This is suggested by the way we treat the measurement result α by summing over it if we do not know the result. We can always imagine another agency; more capable than ourselves; who has access to more fine grained information than what we have. Before the measurement this person knows the particular eigenstate $|e_k\rangle$ that the environment is in but reports to us only the mixed state σ and then after the measurement the person knows which state $|f_{\alpha j}\rangle$ the environment ends up in. In other words he or she know the the result αj of the fine grained measurement in the basis $|f_{\alpha j}\rangle$, but reports to us only the subspace S_α corresponding to the result α. The post measurement state ascribed by this agency to Q is

$$\rho_{\alpha jk} = \frac{A_{\alpha jk}\rho A^\dagger_{\alpha jk}}{p_{\alpha jk}} = \mathcal{A}_{\alpha jk}(\rho),$$

with

$$p_{\alpha jk} = \operatorname{tr}\Big(\mathcal{A}_{\alpha jk}(\rho)\Big) = \operatorname{tr}(\rho A^\dagger_{\alpha jk} A_{\alpha jk}),$$

being the probability associated with the initial state k and the measurement result αj. Knowing the value of α but not knowing j or k we assign a post measurement state that averages over these indices. Given α we have

$$\rho_\alpha = \sum_{j,k} p_{jk|\alpha}\rho_{\alpha jk}.$$

Using Bayes theorem, we have

$$p_{jk|\alpha} = \frac{p_{\alpha jk}}{p_\alpha}.$$

So we have

$$\rho_\alpha = \frac{1}{p_\alpha}\sum_{j,k} p_{\alpha jk}\rho_{\alpha jk} = \frac{1}{p_\alpha}\sum_{j,k} A_{\alpha jk}\rho A^\dagger_{\alpha jk} = \frac{1}{p_\alpha}\mathcal{A}_\alpha(\rho),$$

where

$$p_\alpha = \sum_{j,k} p_{\alpha jk} = \operatorname{tr}\Big(\rho \sum_{j,k} A^\dagger_{\alpha jk} A_{\alpha jk}\Big)$$

Note that even if the conditional and joint probabilities enter the discussion when we are trying to relate the post measurement states of the two agencies, everything about the probabilities disappear from the operations themselves. To construct the operation corresponding to the coarse grained data we just sum over the the fine-grained data that we do not have:

$$\mathcal{A}_\alpha = \sum_{j,k} \mathcal{A}_{\alpha jk}.$$

Although there is a physical difference between the two kinds of potentially available fine-grained information symbolized by the indices j and k, there is no mathematical difference between them, so we can combine them into a single index j in future discussions.