PHY 4105: Quantum Information Theory
Lecture 18

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VonNeumann measurements

A vonNeumann measurement on a system can be thought of in terms of the measurement model as a measurement whose Kraus operators are orthogonal projectors Π_α (not necessarily one dimensional). Since orthogonal projectors commute with each other we can simultaneously diagonalize them in a basis and write,

$$\Pi_\alpha = \sum_j |e_{\alpha j}\rangle\langle e_{\alpha j}|.$$

The POVM elements are the same as the Kraus operators themselves since

$$E_\alpha = \Pi_\alpha \Pi_\alpha^\dagger = \Pi_\alpha.$$

The probability for an outcome α is $p_\alpha = \operatorname{tr}(\Pi_\alpha \rho)$ and the corresponding post measurement state is

$$\rho_\alpha = \frac{\Pi_\alpha \rho \Pi_\alpha}{p_\alpha} = \frac{1}{p_\alpha} \sum_{j,k} |e_{\alpha j}\rangle\langle e_{\alpha j}|\rho|e_{\alpha k}\rangle\langle e_{\alpha k}|.$$

If we do not know the result of the measurement the post measurement state is

$$\rho' = \sum_\alpha p_\alpha \rho_\alpha = \sum_{\alpha,j,k} |e_{\alpha j}\rangle\langle e_{\alpha j}|\rho|e_{\alpha k}\rangle\langle e_{\alpha k}|.$$

The measurement destroys the coherence between each subspace but retains the coherence inside each subspace.

Now suppose that the Kraus operators are one dimensional projectors $\Pi_{\alpha j} = |e_{\alpha j}\rangle\langle e_{\alpha j}|$. These Kraus operators also give rise to the same POVM element,

$$E_\alpha = \sum_j \Pi_{\alpha j} \Pi_{\alpha j}^\dagger = \sum_j \Pi_{\alpha j} = \Pi_\alpha.$$

This means that the measurement statistics are the same for these fine grained Kraus operators as for the coarse grained Kraus operators Π_α, but the post measurement states are different. For the one dimensional projectors, the post measurement state corresponding to the outcome α is

$$\rho_\alpha = \frac{1}{p_\alpha} \sum_j \Pi_{\alpha j} \rho \pi_{\alpha j} = \frac{1}{p_\alpha} \sum_j |e_j\rangle\langle e_j|\rho|e_j\rangle\langle e_j|.$$

This means that the fine grained measurement removes all the coherences within the subspace α as well.

The things to be noted here is that a POVM element can correspond to many different quantum operations. Measurement statistics do not specify the post measurement state. The second point is that the finer-grained the Kraus operators underneath a given POVM elements, the more decoherent the measurement is.

Kraus representation theorem

Given a set of superoperators with Kraus decompositions,

$$\mathcal{A}_\alpha(\rho) = \sum_j A_{\alpha j} \rho A_{\alpha j}^\dagger,$$

where the Kraus operators satisfy the completeness relation,

$$\sum_{\alpha,j} A_{\alpha j}^\dagger A_{\alpha j} = \mathbb{1},$$

there exists an ancilla A with (pure) initial state $\langle e_0 || e_0 \rangle$, a joint unitary operators U on QA and orthogonal projectors P_α on A such that

$$\mathcal{A}_\alpha(\rho) = \operatorname{tr}_A\big(P_\alpha U \rho \otimes |e_0\rangle\langle e_0| U^\dagger\big).$$