PHY 4105: Quantum Information Theory
Lecture 19

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Dynamical maps

We now look at the most general dynamics possible for a quantum system which would included as special cases all the dynamics and transformations we have seen so far including unitary dynamics, measurement models etc. This gives us an abstract characterization of quantum operations in the same spirit as as we had done when we generalized projective measurements into POVMs.

We want to describe general quantum dynamics which has as input a quantum system Q in an input state ρ and that can produce one of many possible outputs labelled by α. Given an input ρ the outcome α happens with probability $p_{\alpha|\rho}$, and the state of the system Q conditioned on the outcome α is ρ_α. Let us consider a map $\mathcal{A}_\alpha$, not assumed to be linear, that takes in the state ρ and outputs ρ_α with the correct probability $p_{\alpha|\rho}$ in a manner familiar to us:

$$p_{\alpha|\rho} = \mathrm{tr}\Big(\mathcal{A}_\alpha(\rho)\Big) \quad \text{and} \quad \rho_\alpha = \frac{\mathcal{A}_\alpha(\rho)}{p_{\alpha|\rho}}.$$

This kind of map is trace decreasing since $\mathcal{A}_\alpha(\rho)$ has trace less than one. If there is only one outcome which happens with probability one, then we can forget about the index α and write the output state as

$$\rho' = \mathcal{A}(\rho).$$

This sort of map is trace preserving. We now argue that $\mathcal{A}_\alpha$ has to be convex linear; i.e.

$$\mathcal{A}_\alpha\big(\lambda\rho_1 + (1-\lambda)\rho_2\big) = \lambda\mathcal{A}_\alpha(\rho_1) + (1-\lambda)\mathcal{A}_\alpha(\rho_2), \quad 0 \leq \lambda \leq 1.$$

The argument proceeds in two stages. First we argue that the traces of both sides of the equation above has to be equal before arguing that the equation above itself has to be true. Imagine that the input state is ρ_1 with probability p_1 and it is ρ_2 with probability p_2. Thus the input state is a mixture,

$$\rho = p_1\rho_1 + p_2\rho_2.$$

The probability for the outcome α can be written in two ways. One is just the definition of the map and all we do is to let in act on the mixed initial state:

$$p_{\alpha|\rho} = \mathrm{tr}\big[\mathcal{A}_\alpha(\rho)\big] = \mathrm{tr}\big[\mathcal{A}_\alpha(p_1\rho_1 + p_2\rho_2)\big].$$

The second way of writing the probability comes from the rules of probability theory:

$$p_{\alpha|\rho} = p_1 p_{\alpha|\rho_1} + p_2 p_{\alpha|\rho_2} = p_1\,\mathrm{tr}\big[\mathcal{A}_\alpha(\rho_1)\big] + p_2\big[\mathcal{A}_\alpha(\rho_2)\big].$$

Equating the two expressions for $p_{\alpha|\rho}$ we get that the trace of both sides of the convex linearity condition we wrote above have to the equal:

$$\mathrm{tr}\big[\mathcal{A}_\alpha(p_1\rho_1 + p_2\rho_2)\big] = p_1\,\mathrm{tr}\big[\mathcal{A}_\alpha(\rho_1)\big] + p_2\big[\mathcal{A}_\alpha(\rho_2)\big].$$

We get the full condition for convex linearity by writing the output state ρ_α in two different ways in a similar manner. The first ways is to say that is we do not know which one of the two the initial state is, we apply the dynamics to the input mixed state and compute the output state:

$$\rho_\alpha = \frac{\mathcal{A}_\alpha(\rho)}{p_{\alpha|\rho}} = \frac{\mathcal{A}_\alpha(p_1\rho_1 + p_2\rho_2)}{p_{\alpha|\rho}}.$$

What we have is mixing first, followed by dynamics.

The second way of writing the output state comes from arguing that the mixing can follow the dynamics and we should get the same result. On the two possible input states, we get the outputs

$$\frac{\mathcal{A}_\alpha(\rho_1)}{p_{\alpha|\rho_1}} \qquad \text{and} \qquad \frac{\mathcal{A}_\alpha(\rho_2)}{p_{\alpha|\rho_2}}.$$

We should be able to mix these outputs and we should get the same output state as the map acting on the mixed initial state since in both cases what is happening is that we are not aware of which one of two possible states the system is in (both initially and finally). We cannot use the same input probabilities p_1 and p_2 to mix the outputs because at this point we know something more about the system in that we know that the outcome α has happened. So we have to update the mixing probabilities to $p_{\rho_1|\alpha}$ and $p_{\rho_2|\alpha}$. So we have the second way of writing ρ_α as

$$\rho_\alpha = p_{\rho_1|\alpha}\frac{\mathcal{A}_\alpha(\rho_1)}{p_{\alpha|\rho_1}} + p_{\rho_2|\alpha}\frac{\mathcal{A}_\alpha(\rho_2)}{p_{\alpha|\rho_2}}.$$

The updated probabilities comes from Bayes theorem:

$$p_{\rho_j,\alpha} = p_{\rho_j|\alpha}p_\alpha = p_{\rho_j|\alpha}p_{\alpha|\rho} = p_{\alpha|\rho_j}p_{\rho_j} = p_{\alpha|\rho_j}p_j.$$

So we have

$$p_{\rho_1|\alpha} = \frac{p_{\alpha|\rho_1}p_1}{p_{\alpha|\rho}} \qquad \text{and} \qquad p_{\rho_2|\alpha} = \frac{p_{\alpha|\rho_2}p_2}{p_{\alpha|\rho}}.$$

Using these updated probabilities in the second way of writing ρ_α we get

$$\rho_\alpha = \frac{1}{p_{\alpha|\rho}}\big[p_1\mathcal{A}_\alpha(\rho_1) + p_2\mathcal{A}_\alpha(\rho_2)\big].$$

Equating the two expressions for the final state we get the convex linearity of the map.

The convex linearity lets us extend the map to the set of all operators, since all density operators can be written as convex linear combinations of the extremal states.

So far we have three conditions for our map $\mathcal{A}$ (we drop the index α since it is not pertinent to our discussions any more)

- *Condition 1*: $\mathcal{A}$ is a map from trace-one positive operators (density operators) to positive operators.

- *Conditions 2*: $\mathcal{A}$ is trace decreasing. i.e. $\mathrm{tr}(\mathcal{A}(\rho)) \leq 1$ for all density operators ρ. Trace preserving dynamics is a special case.

- *Condition 3*: $\mathcal{A}$ is convex linear:

$$\mathcal{A}_\alpha\big(\lambda\rho_1 + (1-\lambda)\rho_2\big) = \lambda\mathcal{A}_\alpha(\rho_1) + (1-\lambda)\mathcal{A}_\alpha(\rho_2), \quad 0 \leq \lambda \leq 1.$$

  This lets us extend $\mathcal{A}$ to a super operator (an operator on all operators).

It is very easy to show that any map with a Kraus decomposition

$$\mathcal{A}(\rho) = \sum_j A_j \rho A_j^\dagger,$$

satisfies the three conditions above. However these three conditions are not sufficient to characterize a quantum operation. For instance consider the Transpose super operator $\mathbb{T}$ which outputs the transpose of the input state with respect to some basis set $\{|e_k\rangle\}$:

$$\mathbb{T}(\rho) = \sum_{j,k} \rho_{kj} |e_j\rangle\langle e_k| \qquad \Leftrightarrow \qquad \mathbb{T} = \sum_{j,k} |e_j\rangle\langle e_k| \odot |e_j\rangle\langle e_j|.$$

We have used the "odor" symbol $\odot$ as a placeholder to indicate the position where the state on which the super operator acts on. This placeholder lets us talk in the abstract about the map without specifying the state its acts on. It turns out that $\mathbb{T}$ is not a quantum operation because it takes positive operators to those that are not positive. This means that we need an additional condition on the maps to make them quantum operations. We will discuss this additional condition first before showing that $\mathbb{T}$ indeed is not a valid operation even though it stands for physically useful transformations like time reversal.

The additional condition can be motivated physically in the following way. Suppose that R is a "reference system" that, though it does not itself take part in the dynamics, cannot be neglected because the initial state ρ of Q is the marginal density operator of a joint state ρ_{RQ}. This certainly being one of the ways to get a density operator for Q, we can't avoid thinking about this situation. We want the map $\mathbb{1}_R \otimes \mathcal{A}$ where $\mathbb{1}_R$ is the unit superoperator acting on R, to be a suitable quantum dynamics, which means that it must take joint states ρ_{RQ} to positive operators, which can be normalized to be output density operators. This requirement holds trivially for a quantum operation, because the operators $\mathbb{1}_R \otimes A_j$ are a Kraus decomposition for the extended operation

$$(\mathbb{1}_R \otimes \mathcal{A})(\rho_{RQ}) = \sum_j (\mathbb{1}_R \otimes A_j)\rho_{RQ}(\mathbb{1}_R \otimes A_j)^\dagger \geq 0.$$

Thus we can add an additional requirement, which strengthens condition 1, to our list of conditions for a map to describe a quantum dynamics:

- *Condition 4*: $(\mathbb{1}_R \otimes \mathcal{A})(\rho_{RQ}) \geq 0$ for all joint density operators ρ_{RQ} of Q and reference systems R of arbitrary dimension. Such a map is said to be *completely positive.*

Our objective now is to show that any map that satisfies conditions 1 through 4 is a quantum operation. Before turning to that task, however, let us first see what goes wrong with an apparently satisfactory map like the transposition superoperator. We want to consider the map $\mathbb{1}_R \otimes \mathbb{T}$, which is called the partial transposition superoperator, because it transposes matrix elements in system Q only. Let us suppose R has the same dimension as Q, and let us apply the partial transposition superoperator to an (unnormalized) maximally entangled state

$$|\Psi\rangle = \sum_j |f_j, e_j\rangle,$$

where $|f_j\rangle$ form an orthonormal basis for R and the vectors $|e_j\rangle$ form an orthonormal basis for Q. Partially transposing Q gives

$$\begin{aligned}
\mathbb{1}_R \otimes \mathbb{T}(|\Psi\rangle\langle\Psi|) &= \mathbb{1}_R \otimes \mathbb{T}\Big(\sum_{j,k} |f_j\rangle\langle f_k| \otimes |e_j\rangle\langle e_k|\Big) \\
&= \sum_{j,k} \mathbb{1}_R(|f_j\rangle\langle f_k|) \otimes \mathbb{T}(|e_j\rangle\langle e_k|) \\
&= \sum_{j,k} |f_j\rangle\langle f_k| \otimes |e_k\rangle\langle e_j| \\
&= \sum_{j,k} |f_j, e_k\rangle\langle f_k, e_j|.
\end{aligned}$$

It is easy to see that the normalized eigenvectors for the $\mathbb{1}_R \otimes \mathbb{T}$ operator are the states $|f_j, e_j\rangle$ for $j = 1, \ldots, D$ and the states

$$\frac{1}{\sqrt{2}}(|f_j, e_k\rangle \pm |f_k, e_j\rangle),$$

for all pairs of indices. The states $|f_j, e_j\rangle$ have eigenvalues $+1$ but the remaining states have eigenvalue ± 1 showing that the operator is not positive (it is actually unitary). This shows that $\mathbb{T}$ cannot have a Kraus decomposition.

This example suggests that the problem with superoperators that are positive, but not completely positive is that, when extended to R, they don't map all entangled states to positive operators, as we would like them to. Indeed, as we now show, the general requirements for complete positivity follow from considering only one kind of reference system, one whose dimension is the same as the dimension of Q, and only one kind of joint state, the maximally entangled state. All we need to consider is how the extended map $\mathbb{1}_R \otimes \mathbb{T}$ acts on $|\Psi\rangle\langle\Psi|$, i.e., the following operator on RQ:

$$\mathbb{1}_R \otimes \mathcal{A}(|\Psi\rangle\langle\Psi|).$$

It can be shown (we will not work through the proof here) that completely positive superoperators that satisfy the four conditions that we laid out, has a Kraus decomposition of the form

$$\mathcal{A} = \sum_\alpha A_\alpha \odot A_\alpha^\dagger.$$

We can show (using Kraus decomposition theorem) that any quantum operation has a Kraus decomposition with no more than D^2 Kraus operators. We also can show that two sets of Kraus operators $\{A_\alpha\}$ and $\{B_\alpha\}$ give rise to the same completely positive superoperator if an only if they are related by a unitary matrix $V_{\alpha\beta}$, i.e.,

$$B_\beta = \sum_\alpha V_{\alpha\beta} A_\alpha.$$

An example will show the importance of this decomposition theorem. Consider a von Neumann measurement in the basis $|e_j\rangle$. If we forget the result of the measurement, the trace-preserving operation that describes the process is

$$\mathcal{A} = \sum_{j=1}^{D} |e_j\rangle\langle e_j| \odot |e_j\rangle\langle e_j| = \sum_j P_j \odot P_j.$$

This operation corresponds to writing the input density operator in the basis $|e_j\rangle$ and then setting all the off-diagonal terms to zero. Physically, it is the ultimate decoherence process: it wipes out all the coherence in the basis $|e_j\rangle$ and replaces the input state with the corresponding incoherent mixture of the basis states $|e_j\rangle\langle e_j|$. Though it would seem to have nothing to do with unitary evolutions, we can nonetheless write this operation as a mixture (convex combination) of unitary operators. If we transform the projectors P_j using the unitary matrix

$$V_{kj} = \frac{1}{\sqrt{D}} e^{2\pi i k j/D},$$

we get new Kraus operators,

$$\frac{1}{\sqrt{D}} U_k = \sum_j V_{kj} P_j = \frac{1}{\sqrt{D}} \sum_j e^{2\pi i k j/D} |e_j\rangle\langle e_j|,$$

The operators U_k are clearly unitary written in their eigen decomposition with phases as eigenvalues. The operation becomes

$$\mathcal{A} = \frac{1}{D} \sum_{k=1}^{D} U_k \odot U_k^\dagger.$$

Thinking in terms of this Kraus decomposition, A describes a process where one chooses one of the D unitaries out of a hat and applies it to the system, not knowing which unitary has been chosen - all have equal probability $1/D$.