PHY 4105: Quantum Information Theory
Lecture 20

Anil Shaji
School of Physics, IISER Thiruvananthapuram
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Qubit operations

Let us now look a bit closely at quantum operations on a single qubit and see when they are completely positive and when they are not. We consider only trace preserving quantum operations. A generic operation has $D^4 = 16$ parameters but the trace preservation condition removes 4 of the parameters and we end up with 12 for qubit operations. The operation is specified by what happens to the pure states and so we need to consider only what happens to the surface of the Bloch sphere of states. Let the operation do the following to the four basis matrices we have for the Bloch sphere:

$$\begin{aligned} \mathcal{A}(\mathbb{1}) &= a\mathbb{1} + \vec{c}\cdot\vec{\sigma}, \\ \mathcal{A}(\sigma_j) &= b_j\mathbb{1} + \sum_k \sigma_k M_{kj}, \end{aligned}$$

so

$$\mathcal{A}(\rho) = \mathcal{A}\bigg(\frac{1}{2}(\mathbb{1} + \vec{S}\cdot\vec{\sigma})\bigg) = \frac{1}{2}\bigg((a + \vec{S}\cdot\vec{b})\mathbb{1} + \vec{c}\cdot\vec{\sigma} + \sum_{j,k}\sigma_k M_{kj} S_j\bigg).$$

The trace preservation condition (irrespective of $\vec{S}$) means that $a = 1$ and $\vec{b} = 0$ (4 parameters fixed). Rewriting $\sum_{j,k}\sigma_k M_{kj} S_j = \vec{\sigma}\cdot M\vec{S}$ we have

$$\mathcal{A}(\rho) = \frac{1}{2}\Big(\mathbb{1} + \vec{\sigma}\cdot(M\vec{S} + \vec{c})\Big) = \frac{1}{2}(\mathbb{1} + \vec{S}'\cdot\vec{\sigma}).$$

So we see that the most general transformation of the Bloch sphere is an affine transformation in which the Bloch vector is transformed into

$$\vec{S}' = M\vec{S} + \vec{c}.$$

Let us now look at the (real) transformation M a bit more closely by doing a polar decomposition of the matrix as

$$M = O\sqrt{M^T M}.$$

$\sqrt{M^T M}$ is a real symmetric matrix which can be decomposed as

$$\sqrt{M^T M} = R_1^T D R_1,$$

where R_1 is a rotation and D is a diagonal matrix. The orthogonal transformation O is a combination of a rotation and a parity operation in general. So we have

$$M = RPR_1^T DR_1,$$

where we choose the parity operation to be a standard one involving, say, the reflection about the xy-plane. Using

$$PRP = R', \qquad RPR_1^T = RPR_1^T PP = R_2P,$$

using $PP = \mathbb{1}$. So we have

$$M = R_2PDR_1 = R_2TR_1,$$

In other words the homogenous part of the most general trace preserving transformation on the state space of a single qubit involves first a rotation of the Bloch sphere, followed by a contraction (and possible reflection) along the (rotated) x, y and z axes converting the sphere into an ellipsoid followed by a further rotation of the ellipsoid. The in homogenous part ($\vec{c}$) shifts the entire ellipsoid still keeping it inside the Bloch sphere (for complete positivity). In other words

$$\vec{S}' = R_2TR_1\vec{S} + \vec{c} = R_2(TR_1\vec{S} + \vec{d}), \qquad \vec{d} = R_2^T\vec{c}.$$

The two rotations carry three parameters each, $T = t_j\delta_{ij}$ and $\vec{c}$ also carry three parameters each, making the total up to 12.

What we have seen is how the general operation looks on the Bloch sphere. How does it get implemented in the Hilbert space of the qubit. This is easily written down using the identity we have already established,

$$U_R\vec{\sigma}U_R^\dagger = R\sigma.$$

Let us first separate out the contraction and displacement part of the maps as

$$\mathcal{B}(\rho) = \frac{1}{2}(\mathbb{1} + \vec{\sigma}\cdot(T\vec{S} + \vec{d})).$$

Then it turns out that

$$\mathcal{A}(A) = U_{R_2}\mathcal{B}(U_{R_1}\rho U_{R_1}^\dagger)U_{R_2}^\dagger.$$

The initial and final rotations becomes unitaries as expected.

We will not deal with the conditions for complete positivity here since that required a few more items of notation and some more arguments that is not possible in the limited time. Physically however what it means is that $\mathcal{A}$ should be such that after all the rotations, displacements and contractions, the final ellipsoid of states (set of states on to which the Bloch sphere gets mapped into) has to lie inside the original Bloch sphere.

Examples

1. Projection of Bloch sphere onto the z axis:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Kraus operators are $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$:

$$\begin{aligned} \mathcal{A}(X) &= P_0XP_0 + P_1XP_1 &= 0 \\ \mathcal{A}(Y) &= P_0YP_0 + P_1YP_1 &= 0 \\ \mathcal{A}(Z) &= P_0ZP_0 + P_1ZP_1 &= Z \end{aligned}$$

This T corresponds to measurement in the Z basis.

2. Stochastic flip operation: The Kraus operators are $\sqrt{p}\mathbb{1}$, $\sqrt{1-p}\sigma_j$. If $\sigma_j = X$ then we have the bit flip operation and if $\sigma_j = Z$ we have phase flip. If $\sigma_j = Y = -iZX = iXZ$ then we have both phase and bit flip. The map is

$$\mathcal{A} = p\mathbb{1} \odot \mathbb{1} + (1-p)\sigma_j \odot \sigma_j.$$

Let us consider the case where $\sigma_j = X$ giving us bit flip, so that

$$\mathcal{A} = p\mathbb{1} \odot \mathbb{1} + (1-p)X \odot X.$$

We have $\mathcal{A}(\mathbb{1}) = \mathbb{1} \Rightarrow \vec{c} = 0$. We also have

$$\begin{aligned} \mathcal{A}(X) &= X \\ \mathcal{A}(Y) &= (2p-1)Y \\ \mathcal{A}(Z) &= (2p-1)Z \end{aligned}$$

So in the Bloch sphere representation we have

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2p-1 & 0 \\ 0 & 0 & 2p-1 \end{pmatrix}.$$

We have contraction about the Z and Y axes. When $p = 1$ we have the identity map. When $p = 1/2$ we have measurement along the X basis and $p = 1$ corresponds to pure bit flip. A similar analysis can be done for phase flip and phase plus bit flip channels. Note that these processes are realistic errors that might occur during a quantum information processing protocol.

3. Depolarizing map: The Kraus operators are

$$\frac{\sqrt{p}}{2}\mathbb{1},\ \frac{\sqrt{p}}{2}X,\ \frac{\sqrt{p}}{2}Y,\ \frac{\sqrt{p}}{2}Z,\ \sqrt{1-p}\mathbb{1}.$$

The map is

$$\mathcal{A} = p\frac{1}{4}(\mathbb{1} \odot \mathbb{1} + X \odot X + Y \odot Y + Z \odot Z) + (1-p)\mathbb{1} \odot \mathbb{1}.$$

Complete depolarization corresponds to any density matrix ρ being reduced the completely mixed state $\mathbb{1}/2$. The action of the above map is

$$\mathcal{A}(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho.$$

So for $p = 1$ we have complete depolarization. We have $\mathcal{A}(\mathbb{1}) = \mathbb{1}$ which means that $\vec{c} = 0$. So we have the Bloch sphere representation,

$$\mathcal{A}(\sigma_j) = (1-p)\sigma_j \quad \Leftrightarrow \quad T = (1-p)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Amplitude damping (spontaneous decay from $|0\rangle$ to $|1\rangle$)
We start with a measurement model for amplitude damping in which we have a combined unitary for the qubit and its environment, U which does the following transformations:

$$\begin{aligned} |1\rangle|\text{Env ground state}\rangle &\rightarrow |1\rangle|\text{Env ground state}\rangle \\ |0\rangle|\text{Env ground state}\rangle &\rightarrow \sqrt{1-\gamma}|0\rangle|\text{Env ground state}\rangle + \sqrt{\gamma}|1\rangle|\text{Decay products}\rangle \end{aligned}$$

Note that the two final states are mutually orthogonal and γ gives the probability for decay. The initial state can be written as

$$\begin{aligned} \rho \otimes |\text{egs}\rangle\langle\text{egs}| &= \rho_{11}|1\rangle|\text{egs}\rangle\langle 1|\langle\text{egs}| + \rho_{10}|1\rangle|\text{egs}\rangle\langle 0|\langle\text{egs}| \\ &\quad + \rho_{01}|0\rangle|\text{egs}\rangle\langle 1|\langle\text{egs}| + \rho_{00}|0\rangle|\text{egs}\rangle\langle 0|\langle\text{egs}|. \end{aligned}$$

The combined evolution of the qubit and the environment leads to the state,

$$\begin{aligned} \mathcal{R}_{qe} &= \rho_{11}|1\rangle|\text{egs}\rangle\langle 1|\langle\text{egs}| \\ &\quad + \rho_{10}|1\rangle|\text{egs}\rangle(\sqrt{1-\gamma}\langle 0|\langle\text{egs}| + \sqrt{\gamma}\langle 1|\langle\text{dp}|) \\ &\quad + \rho_{01}(\sqrt{1-\gamma}|0\rangle|\text{egs}\rangle + \sqrt{\gamma}|1\rangle\langle\text{dp}|)\langle 1|\langle\text{egs}| \\ &\quad + \rho_{00}(\sqrt{1-\gamma}|0\rangle|\text{egs}\rangle + \sqrt{\gamma}|1\rangle\langle\text{dp}|)(\sqrt{1-\gamma}\langle 0|\langle\text{egs}| + \sqrt{\gamma}\langle 1|\langle\text{dp}|). \end{aligned}$$

The map $\mathcal{A}(\rho)$ is obtained by tracing out the environment:

$$\begin{aligned} \mathcal{A}(\rho) &= \rho_{11}|1\rangle\langle 1| + \sqrt{1-\gamma}\rho_{10}|1\rangle\langle 0| + \sqrt{1-\gamma}\rho_{01}|0\rangle\langle 1| + \rho_{00}\big[(1-\gamma)|0\rangle\langle 0| + \gamma|1\rangle\langle 1|\big], \\ &= \big[|1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0|\big]\rho\big[|1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0|\big] + \sqrt{\gamma}|1\rangle\langle 0|\rho|0\rangle\langle 1|\sqrt{\gamma}. \end{aligned}$$

So we have the Kraus operators

$$\begin{aligned} A_1 = & \ |1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0| \ \leftrightarrow \begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \\ A_2 = & \ \sqrt{\gamma}|1\rangle\langle 0| \ \leftrightarrow \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}. \end{aligned}$$

So we have

$$\mathcal{A} = A_1 \odot A_1^\dagger + A_2 \odot A_2^\dagger.$$

To find the Bloch sphere description we consider,

$$\mathcal{A}(\mathbb{1}) = A_1 A_1^\dagger + A_2 A_2^\dagger = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} = 1 - \gamma Z$$

This means that $\vec{c} = -\gamma \vec{e}_z$. We also have

$$\begin{aligned} \mathcal{A}(X) &= \sqrt{1-\gamma} X \\ \mathcal{A}(Y) &= \sqrt{1-\gamma} Y \\ \mathcal{A}(Z) &= (1-\gamma) Z, \end{aligned}$$

So that

$$T = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}.$$

We can write a differential equation for the decay process by setting $\gamma = Rdt$. So we have

$$\vec{S}' = T\vec{S} + \vec{c} = \begin{pmatrix} 1-\frac{1}{2}Rdt & 0 & 0 \\ 0 & 1-\frac{1}{2}Rdt & 0 \\ 0 & 0 & 1-Rdt \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} - Rdt\hat{z}.$$

This gives us,

$$\frac{dS_x}{dt} = -\frac{1}{2}RS_x, \qquad \frac{dS_y}{dt} = -\frac{1}{2}RS_y, \qquad \frac{dS_z}{dt} = -RS_z - R.$$

The solutions to these equations have the form

$$S_x(t) = e^{-Rt/2}S_x(0), \qquad S_y(t) = e^{-Rt/2}S_y(0), \qquad S_z(t) = -1 + e^{-Rt}(1 + S_z(0)).$$