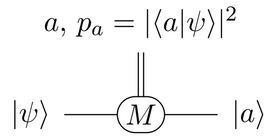
PHY 4105: Quantum Information Theory
Lecture 23

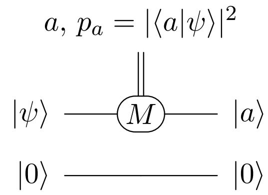
Anil Shaji
School of Physics, IISER Thiruvananthapuram
(Dated: November 4, 2013)

Useful circuits

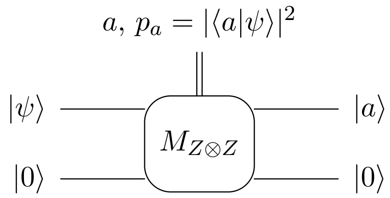
Now that we are just done with introducing measurements into the circuit model, let us look at how the measurement model that we have talked about looks in the circuit language. We have represented a measurement as



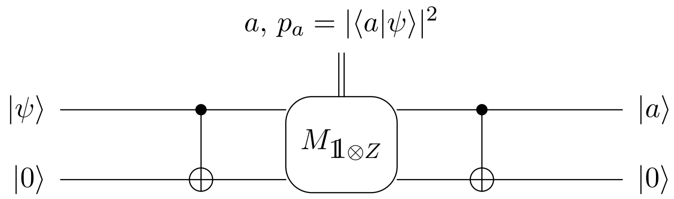
Let us now introduce an ancilla, which for the moment does nothing:



We now expand the measurement to measure in the Z basis on the ancilla as well, which for the moment does nothing again:



As such we are doing a partial measurement since we are recording the probability of finding a for the first qubit still and not looking at the state of the ancilla. Now consider the following circuit:



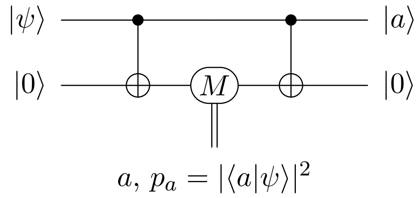
The initial state of the two qubits is

$$(\langle 0|\psi\rangle|0\rangle + \langle 1|\psi\rangle|1\rangle) \otimes |0\rangle.$$

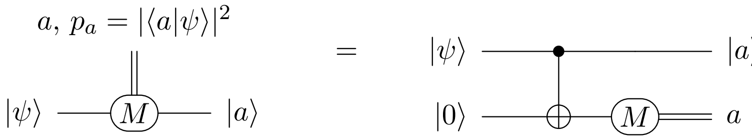
After the first CNOT we get

$$\langle 0|\psi\rangle|00\rangle + \langle 1|\psi\rangle|11\rangle.$$

So measuring the outcomes $|0\rangle$ and $|1\rangle$ on the ancilla qubit using the Z basis measurement gives the respective outcomes with the same probabilities as the Z basis measurement on the first qubit. Since the measurement of the identity operator on the first qubit really does nothing, we can rewrite the above circuit as



If we do not care about the post measurement state of the second qubit, we can as well remove the second CNOT operation. So we have



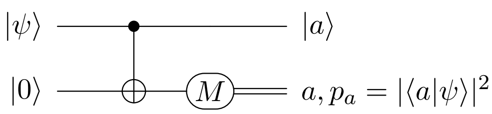
The CNOT is the canonical measurement gate. We can develop general measurement models including POVMs using the Neumark extension and then following the procedure above.

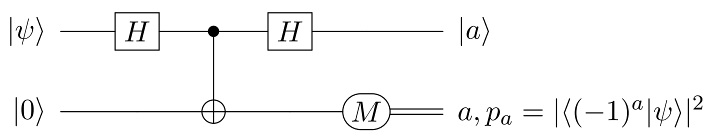
1. *A four outcome POVM*

We have seen a four outcome POVM before where we measure a qubit in the z-basis or the x-basis with equal probability. The corresponding Kraus operators and POVM elements (we have seen before) are

$$A_{00} = \frac{1}{\sqrt{2}}|0\rangle\langle 0| \qquad E_{00} = \tfrac{1}{2}|0\rangle\langle 0|$$
$$A_{01} = \frac{1}{\sqrt{2}}|1\rangle\langle 1| \qquad E_{00} = \tfrac{1}{2}|1\rangle\langle 1|$$
$$A_{10} = \frac{1}{\sqrt{2}}|+\rangle\langle +| \qquad E_{00} = \tfrac{1}{2}|+\rangle\langle +|$$
$$A_{11} = \frac{1}{\sqrt{2}}|-\rangle\langle -| \qquad E_{00} = \tfrac{1}{2}|-\rangle\langle -|$$

The two circuits corresponding to these two measurements are





In the first circuit that corresponds to measurement in the computational basis, the initial state if $|\psi\rangle \otimes |0\rangle$. After the CNOT we get $\langle 0|\psi\rangle|00\rangle + \langle 1|\psi\rangle|11\rangle$. So the measurement on the second qubit realizes one of the terms in the superposition and effects a measurement in the computational basis.

In the second circuit, again the initial state is $|\psi\rangle \otimes |0\rangle$. After the first HADAMARD we get the state

$$\big(|0\rangle\langle +|\psi\rangle + |1\rangle\langle -|\psi\rangle\big) \otimes |0\rangle.$$

This is obtained by using

$$|\psi\rangle = |0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle = |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle,$$

and applying the HADAMARD on the second form and then using $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$. After the CNOT we have

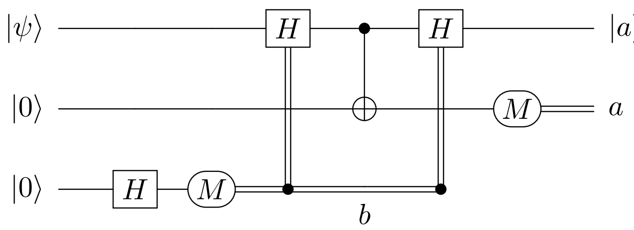
$$\langle +|\psi\rangle|00\rangle + \langle -|\psi\rangle|11\rangle.$$

The next Hadamard makes the state

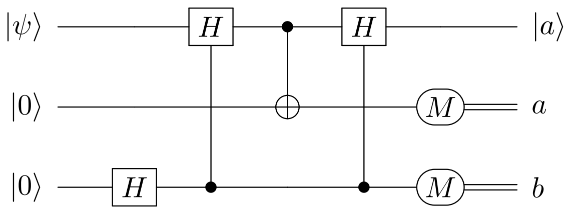
$$\langle +|\psi\rangle|+0\rangle + \langle -|\psi\rangle|-1\rangle.$$

Now a measurement in the computational basis on the second qubit gives the post measurement state $|\pm\rangle$ for the top qubit with probabilities $|\langle \pm|\psi\rangle|^2$.

As mentioned before, what we are doing here is first flipping a coin and depending on the outcome choosing to measure X or Z on the qubit. We can make a third qubit do the coin flipping and write the circuit as



We can move the measurement through the controls using the principle of deferred measurement and we get a coherent version of the circuit as



After the first Hadamard the state is

$$|\psi\rangle \otimes |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

The controlled H that follows makes the state

$$\frac{1}{\sqrt{2}}(|\psi\rangle \otimes |00\rangle + H|\psi\rangle \otimes |01\rangle).$$

The CNOT transforms this state to

$$\frac{1}{\sqrt{2}}\Big(\langle 0|\psi\rangle|000\rangle + \langle 1|\psi\rangle|110\rangle\Big) + \frac{1}{\sqrt{2}}\Big(\langle +|\psi\rangle|001\rangle + \langle -|\psi\rangle|111\rangle\Big).$$

The last controlled H transforms this state to

$$\frac{1}{\sqrt{2}}\Big(\langle 0|\psi\rangle|000\rangle + \langle 1|\psi\rangle|110\rangle\Big) + \frac{1}{\sqrt{2}}\Big(\langle +|\psi\rangle|{+01}\rangle + \langle -|\psi\rangle|{-11}\rangle\Big).$$

, which can be written as

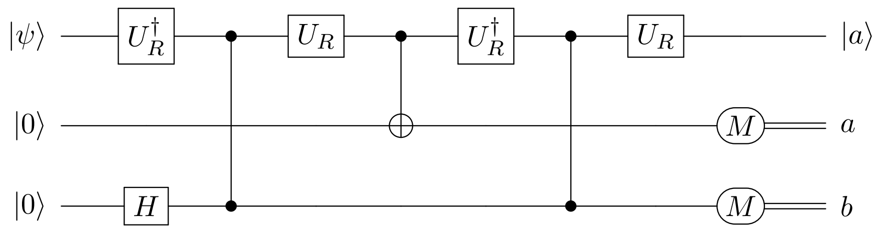
$$\frac{1}{\sqrt{2}}\Big(\langle 0|\psi\rangle|0\rangle\otimes|00\rangle + \langle 1|\psi\rangle|1\rangle\otimes|10\rangle + \langle +|\psi\rangle|+\rangle|01\rangle + \langle -|\psi\rangle|-\rangle|11\rangle\Big),$$

showing explicitly the four post measurement states and their corresponding probabilities.

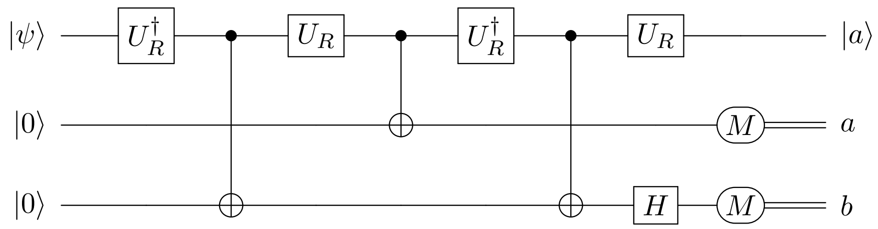
Using a $\pi/4$ rotation about the Y axis that does the transformation

$$U_R Z U_R^\dagger = H,$$

we can "improve" the above circuit as



We can move the HADAMARD on the third qubit all the way to the right as



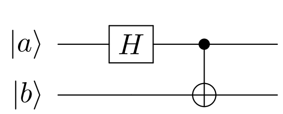
This last form is not at all obvious without using quantum circuit diagrams.

2. *Creating bell states*

The circuit for making the Bell states can be drawn out easily starting from

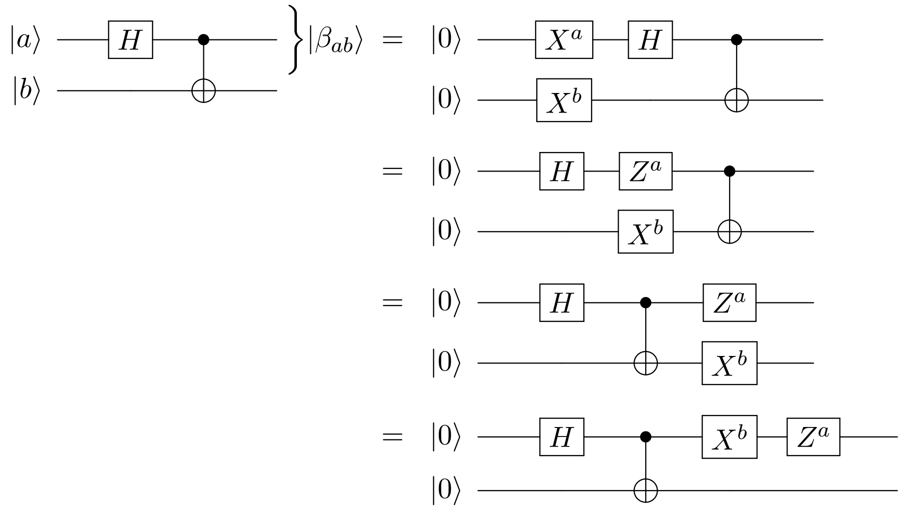
$$|\beta_{ab}\rangle = Z^a X^b \otimes \mathbb{1}|\beta_{00}\rangle = \mathbb{1}\otimes X^b Z^a|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\big(|0b\rangle + (-1)^a|1\bar{b}\rangle\big) = \mathrm{CNOT}(H\otimes\mathbb{1})|a,b\rangle.$$

so we have



$$\Big\}|\beta_{ab}\rangle = \frac{1}{\sqrt{2}}\big(|0b\rangle + (-1)^a|1\bar{b}\rangle\big).$$

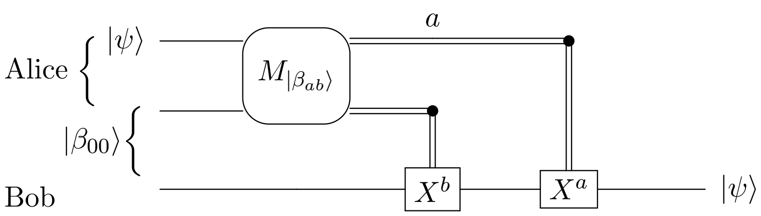
The equivalence between the circuits making the Bell states can be seen from the following:



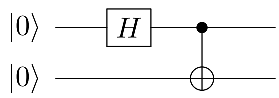
After the CNOT we get the $|\beta_{00}\rangle$ state.

3. *Teleportation circuit*

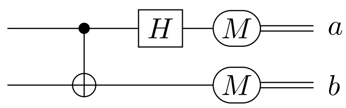
The teleportation protocol we have seen before can be written as a circuit as below:



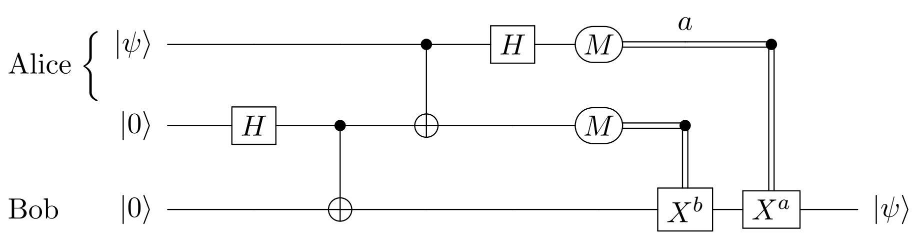
The initial Bell state is made by the circuit



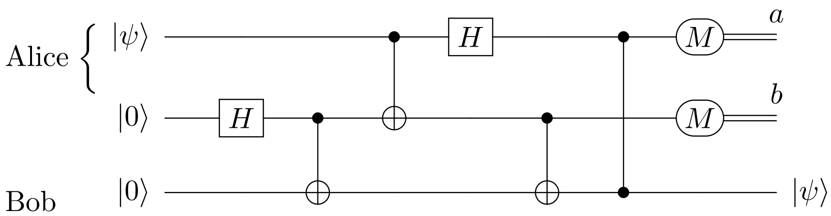
Similarly a measurement in the bell basis is achieved by the circuit



So we can put it all together and have the teleportation circuit as



We can refine this circuit by moving the measurements past the control using the principle of deferred measurement:



This is a reversible quantum teleportation circuit. Here the measurements and their results in the end are rather irrelevant. They only, after the fact, tell us the actions taken by the CNOT and CSIGN gates. However the point of teleportation is that Alice and Bob communicate classically. Here there is a direct quantum communication between them in order to make the quantum CNOT and CSIGN gates possible.