PHY 4105: Quantum Information Theory
Lecture 8

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qubits

The quantum analogue of a bit, aptly named as a *qubit* has states described by rays in a two dimensional ($D = 2$) Hilbert space. The fiducial basis or computational basis in the two dimensional vector space is formed by vectors denoted as

$$|0\rangle \qquad \text{and} \qquad |1\rangle.$$

A few physical realizations of qubit include:

1. Spin-1/2 particles: $|0\rangle = |\uparrow\rangle$ and $|1\rangle = |\downarrow\rangle$
2. Polarization of a photon: $|0\rangle = |R\rangle$ and $|1\rangle = |L\rangle$
3. Two states of an atom $|0\rangle = |e\rangle$ and $|1\rangle = |g\rangle$

An arbitrary pure state of a qubit is given by

$$|\psi\rangle = a|0\rangle + b|1\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Using the freedom to choose the overall phase, we can set a to be real and positive and parametrize it with an angle as $a = \cos\theta/2$ while fixing the relative phase between $|0\rangle$ and $|1\rangle$ to be between 0 and 2π we can choose $b = e^{i\phi}\sin\theta/2$, so that a general state is represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = |\vec{n}\rangle = |+1, \vec{n}\rangle,$$

where $\vec{n}$ is the unit vector parametrized by the angles θ and ϕ. The unit vector $-\vec{n}$ is parameterized by $\pi - \theta$ and $\phi + \pi$:

$$|-\vec{n}\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle.$$

We have $\langle\vec{n}|-\vec{n}\rangle = 0$. By parameterizing states of a qubit with two angles, which in turn specify a unit vector in a real three dimensional vector space, we are identifying states with points on the surface of a unit sphere in real space. For instance

$$|0\rangle = |+1, \vec{e}_z\rangle = |\vec{e}_z\rangle \qquad \text{and} \qquad |1\rangle = |-1, \vec{e}_z\rangle = |-\vec{e}_z\rangle.$$

In general $|x\rangle = |(-1)^x, \vec{e}_z\rangle$. The unit sphere of states is called the Bloch sphere when we are dealing with states of a spin-1/2 system while it is called the Poincare sphere when we are dealing with

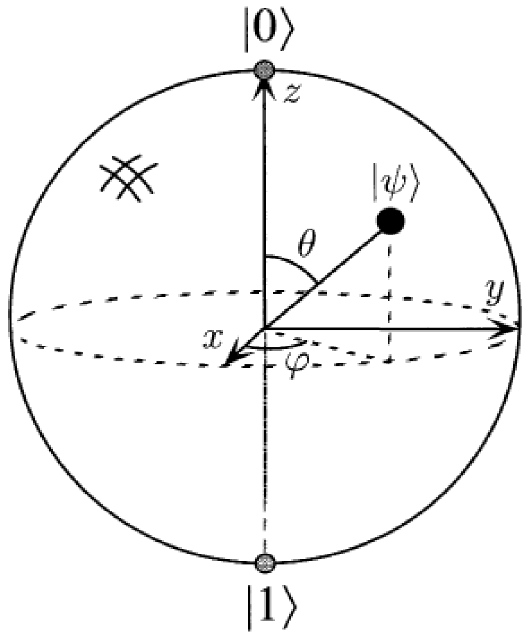

FIG. 1: The Bloch sphere for a qubit

the polarization states of a photon. The Bloch sphere and the location of some of the commonly encountered states on it are shown in the figure below.

The projector along an arbitrary pure state of a qubit is given by

$$\begin{aligned}
P_{\vec{n}} &= |\vec{n}\rangle\langle\vec{n}| \\
&= \cos^2\frac{\theta}{2}|0\rangle\langle 0| + \sin^2\frac{\theta}{2}|1\rangle\langle 1| + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(e^{i\phi}|1\rangle\langle 0| + e^{-i\phi}|0\rangle\langle 1|) \\
&= \frac{1}{2}(1+\cos\theta)|0\rangle\langle 0| + \frac{1}{2}(1-\cos\theta)|1\rangle\langle 1| \\
&\qquad + \frac{1}{2}\sin\theta[(\cos\phi + i\sin\phi)|1\rangle\langle 0| + (\cos\phi - i\sin\phi)|0\rangle\langle 1|] \\
&= \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1| + \cos\theta(|0\rangle\langle 0| - |1\rangle\langle 1|) + \sin\theta\cos\phi(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&\qquad\qquad + \sin\theta\sin\phi(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)\Big].
\end{aligned}$$

We can now identify the following operators,

$$\begin{aligned}
\sigma_x = \sigma_1 = X &= |0\rangle\langle 1| + |1\rangle\langle 0| \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_y = \sigma_2 = Y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_z = \sigma_3 = Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned}$$

and along with the coefficients $n_x = \sin\theta\cos\phi$, $n_y = \sin\theta\sin\phi$ and $n_z = \cos\theta$, we can write

$$P_{\vec{n}} = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma}).$$

The operators $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are called the Pauli spin operators or the Pauli matrices. The operator corresponding to the total spin of a spin-1/2 system is given by

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma},$$

as the name suggests. We will be dealing rather extensively with the Pauli operators through this course and it is therefore worthwhile to enumerate and understand its properties.

1. Hermitian: $\sigma_j = \sigma_j^\dagger$ and Unitary: $\sigma_j\sigma_j^\dagger = \sigma_j^\dagger\sigma_j = \sigma_j^2 = \mathbb{1}$.

2. $\sigma_j\sigma_k = \mathbb{1}\delta_{jk} + i\epsilon_{jkl}\sigma_l$, with repeated indices summed over and ϵ_{jkl} being the antisymmetric symbol. This means that all products of Pauli matrices can be reduced to one of the three matrices or to the unit operator.

$$\begin{aligned} \sigma_1\sigma_2 &= -\sigma_2\sigma_1 &= i\sigma_3, \\ \sigma_2\sigma_3 &= -\sigma_3\sigma_2 &= i\sigma_1, \\ \sigma_3\sigma_1 &= -\sigma_1\sigma_3 &= i\sigma_2. \end{aligned}$$

   From the above we get

$$[\sigma_j,\, \sigma_k] = 2i\epsilon_{jkl}\sigma_l, \qquad [\sigma_j,\, \sigma_k]_+ = \sigma_j\sigma_k + \sigma_k\sigma_j = 2\mathbb{1}\delta_{jk}.$$

   Specifically,

$$[\sigma_1,\, \sigma_2] = 2i\sigma_3, \quad [\sigma_2,\, \sigma_3] = 2i\sigma_1, \quad [\sigma_3,\, \sigma_1] = 2i\sigma_2.$$

3. $\mathrm{tr}(\sigma_j) = \mathrm{tr}(\vec{n}\cdot\vec{\sigma}) = 0$.

4. Orthogonality: $\mathrm{tr}(\sigma_j^\dagger\sigma_k) = \mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}$.

5. The operators $\mathbb{1}$, σ_1, σ_2 and σ_3 form a basis for the 4 dimensional space of operators (2×2 matrices) acting on single qubit states. Any operator can be written as

$$A = A_0\mathbb{1} + A_j\sigma_j = A_0\mathbb{1} + \vec{A}\cdot\vec{\sigma} = A_\alpha\sigma_\alpha, \qquad \alpha = 0, 1, 2, 3, \quad \sigma_0 \equiv \mathbb{1}.$$

   $A^\dagger = A_\alpha^*\sigma_\alpha$ and if A is Hermitian then A_α are real.

6. From the orthogonality condition, $\mathrm{tr}(\sigma_\alpha^\dagger\sigma_\beta) = \mathrm{tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$, we have

$$A = A_\alpha\sigma_\alpha \quad \Leftrightarrow \quad A_\alpha = \frac{1}{2}\,\mathrm{tr}(\sigma_\alpha A).$$

7. If $A = A_\alpha\sigma_\alpha$ and $B = B_\alpha\sigma_\alpha$, then

$$AB = (A_0B_0 + \vec{A}\cdot\vec{B})\mathbb{1} + (A_0\vec{B} + B_0\vec{A} + i\vec{A}\times\vec{B})\cdot\vec{\sigma}.$$

$$\Rightarrow \quad \mathrm{tr}(AB) = 2(A_0B_0 + \vec{A}\cdot\vec{B}) = 2A_\alpha B_\alpha.$$

As a special case

$$(\vec{n}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma}) = \vec{n}\cdot\vec{m}\mathbb{1} + i(\vec{n}\times\vec{m})\cdot\vec{\sigma}.$$

We also have

$$[A,\, B] = 2i(\vec{A}\times\vec{B})\cdot\vec{\sigma}.$$

$$[A,\, A^\dagger] = 2i(\vec{A}\times\vec{A}^*)\cdot\vec{\sigma},$$

So if A is normal, then $\vec{A}\times\vec{A}^* = 0$ and this also means $\vec{A} = e^{i\phi}\vec{S}$, where $\vec{S}$ is a real vector.

8. $P_{\vec{n}} = (\mathbb{1} + \vec{n}\cdot\vec{\sigma})/2$

$$\mathbb{1} = P_{\vec{n}} + P_{-\vec{n}} = |\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|,$$

$$\vec{n}\cdot\vec{\sigma} = P_{\vec{n}} - P_{-\vec{n}} = |\vec{n}\rangle\langle\vec{n}| - |-\vec{n}\rangle\langle-\vec{n}|.$$

The second equation is the eigen-decomposition of $\vec{n}\cdot\vec{\sigma}$.

9. If $A = A_0\mathbb{1} + \vec{A}\cdot\vec{\sigma}$ is Hermitian, then we can define a unit vector $\vec{A}/|\vec{A}|$, giving

$$A = A_0\mathbb{1} + |\vec{A}|\vec{n}\cdot\vec{\sigma} = (A_0 + |\vec{A}|)|\vec{n}\rangle\langle\vec{n}| + (A_0 - |\vec{A}|)|-\vec{n}\rangle\langle-\vec{n}|.$$

This gives the eigenvalues and eigenvectors of A.
If $\vec{A}$ is a normal operator so that $\vec{A} = e^{i\phi}|\vec{A}|\vec{n}$, then $A_0 \pm e^{i\gamma}|\vec{A}|$ are the eigenvalues of the operator.

10. We can define *raising* and *lowering* operators,

$$\begin{aligned}
\sigma_+ &= \frac{1}{2}(\sigma_1 + i\sigma_2) \ = |0\rangle\langle 1| \ \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\sigma_- &= \frac{1}{2}(\sigma_1 - i\sigma_2) \ = |1\rangle\langle 0| \ \leftrightarrow \begin{pmatrix} 0 & 0 \\ 01 & 0 \end{pmatrix}.
\end{aligned}$$

$$\sigma_\pm^2 = 0, \quad \sigma_\pm\sigma_\mp = \frac{1}{2}(\mathbb{1} \pm \sigma_3), \quad \sigma_\pm\sigma_3 = \mp\sigma_\pm, \quad \sigma_3\sigma_\pm = \pm\sigma_\pm.$$

$$[\sigma_\pm,\, \sigma_\mp] = \pm\sigma_3, \quad [\sigma_\pm,\, \sigma_\mp]_+ = \mathbb{1}, \quad [\sigma_\pm,\, \sigma_3] = \mp 2\sigma_\pm, \quad [\sigma_\pm,\, \sigma_3]_+ = 0.$$

11. $e^{i\vec{n}\cdot\vec{\sigma}\gamma} = \mathbb{1}\cos\gamma + i\vec{n}\cdot\vec{\sigma}\sin\gamma.$