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| IISER TVM | Varsha 2013 |
| PHY 4105 | Problem Set 1 (Due on 23/8/2013) |

# 1 Classical Probability

1. A binary experiment is conducted such that $p(0) = q$ and $p(1) = p = 1 - q$. Also $n$ independent trials of this experiment are performed, and we are interested in the random variable $K$ that counts the number of 1's that occur. $K$ has $n + 1$ possible values $0, 1, ..., n$.

   (a) Show that the probability distribution for $K$ is given by

   $$p(k) = \binom{n}{k} p^k q^{n-k}$$

   (b) Find the mean and the variance of the binomial distribution.

   (c) $Stirling's\, formula$ provides an approximation of $n!$ for large $n$:

   $$n! \approx \sqrt{2\pi n}\, n^n e^{-n}$$

   Use this to find an approximate formula for the binomial coefficient. Once you have done this, estimate the value of $k$ that maximizes the probability $p(k)$ for large $n$. What do you find?

# 2 Classical Information

(a) Prove Jensen's inequality:

$$\sum_i p_X(x_i) f(x_i) \geq f\left(\sum_i p_X(x_i) x_i\right)$$

where $X$ is a random variable and $f(x)$ is a convex function.

(b) Prove Fano's inequality:

$$H(p_{err}) + p_{err} \log(|\chi|) \geq H(X|Y)$$

with $X$ a random variable with the probability density function $p_X(x)$ and $|\chi|$ the number of elements in the range of $X$. $Y$ is another random variable related to $X$, with the conditional probability $p_{Y|X}(y|x)$. $p_{err} = \text{Prob}(\hat{X} \neq X)$ is the error when we estimate $X$ based on the observation of $Y$ as $\hat{X} = f(X)$.

# 3 Typical Sequences

A sequence of 16 binary symbols are sent out from a source which produces two independent binary symbols with $p(0) = p$ and $p(1) = 1 - p$ with $p > 0.5$. A typical sequence is defined to have two or fewer symbols of 1.

(a) What is the most probable sequence that can be generated by this source and what is its probability?

(b) What is the number of typical sequences that can be generated by this source?

Assume that one assigns a unique binary codeword for each typical sequence and neglect the non- typical sequences.

(c) If the assigned codewords are all of the same length, find the minimum codeword length required to provide the above set with distinct codewords.

(d) Determine the probability that a sequence is not assigned with a codeword.