PHY 4105: Quantum Information Theory
Lecture 1

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How does one describe the state of a one rupee coin in flight as the umpire tosses it to pretty much decide the fate of a cricket match on a windy and wet morning that is probably going to turn sunny later? One could write down the equations of motion for the center of mass of the coin and then put a body-fixed set of axes on the coin and write down the equations of motion for the Euler angles for the body fixed axes relative to a fixed set of axes to describe how the coin spins as its center of mass moves through the air. One can even add terms for air resistance and forces due to the wind and Coriolis forces and so on. However one will not be able to solve these because the initial conditions determined by the umpire's finger and the exact wind patterns at the moment of tossing are all not known to you.

Instead what really, succinctly captures what we really are interested about regarding the coin is a probability vector of the form

$$p = \begin{pmatrix} p_H \\ p_T \end{pmatrix}.$$

The probability vector is indeed a dynamical quantity just like $x(t)$, $p(t)$, $\omega(t)$ etc of the coin and you can see this clearly if you imagine the the coin falls on a hard surface, starts to spin and then sways to one side and then to the other and so on. As it is swaying you are free to update p_H and p_T to values other than 1/2 each. Also, a bookie and a spot fixer might together know that p_H and p_T are not equal to 1/2 even prior to the coin being tossed.

It is a point of great debate whether to ascribe this description of the "state" of the coin to the coin itself or to the person looking at the coin. The debate goes by various names; Bayesian vs Frequentist approach, ontic versus epistemological description etc. We will see shades of this as we go along although a thorough discussion is way beyond the scope of this course.

Coming back to the coin that is being tossed, in the end two things or "events" can happen. Event E_1 corresponds to the coin coming to rest "heads" up and event E_2 corresponds to the coin coming to rest "tails" up. How do we ascribe probabilities to these events? For that we need to resort to the axioms of probability theory. Before looking into that let us look at the different events that can happen. For this purpose having only two events is not very interesting because really all that can happen is either E_1 or E_2. Let us increase the possibilities a bit further and consider three events E_1, E_2 and E_3. Let us fix notation first by defining the symbols below:

Logical Symbol	**Description**	**Set theoretic description**
$\vee$	OR	$\cup$
$\wedge$	AND	$\cap$
$\neg$	NOT	complement

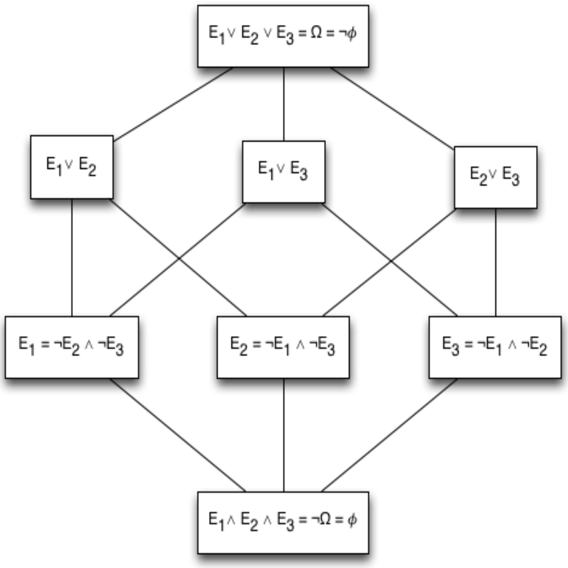

FIG. 1: Boolean lattice of events, propositions and alternatives constructed from three possible events E_1, E_2 and E_3

Starting with three possible events we can construct a Boolean lattice of the possibilities that are available as shown in Fig. 1. Each box represents the different alternatives that can possibly happen. The events E_1, E_2, and E_3 are atomic alternatives

Our task is to assign probabilities to each of these boxes in a consistent manner (assign a measure on the event space). It turns out we can do it using the following axioms.

1. $0 \leq p(E) \leq 1$

2. if E is certain, then $p(E) = 1$

3. Probabilities of mutually exclusive events are additive:

$$p(E_1 \vee E_2) = p(E_1) + p(E_2) \quad \text{if} \quad E_1 \wedge E_2 = \phi$$

4. Bayes's rule

$$p(E_1 \wedge E_2) = p(E_1|E_2)p(E_2).$$

If we consider a certain event E then axioms 2 and 3 together tell us that

$$p(E) + p(\neg E) = p(E \vee \neg E) = p(\Omega) = 1,$$

which is also recognizable as the normalization condition on probabilities.

Using the axioms of probabilities, we can now assign probabilities for a random variable X which can take values $x_1, x_2 \ldots, x_N$ takes on a value x as

$$p_X(x).$$

For two random variables X and Y we can define a joint probability

$$p_{X,Y}(x, y),$$

marginal probabilities,

$$p_X(x) = \sum_Y P_{X,Y}(x, y), \qquad p_Y(y) = \sum_x p_{X,Y}(x, y),$$

and conditional probabilities,

$$p_{X|Y}(x|y)p_Y(y) = p_{X,Y}(x, y) = p_{Y|X}(y|x)p_X(x).$$

We pretty much always will drop the subscript that tells which variable(s) the probability assignment refers to and rather use the arguments to tell us that information.