PHY 4105: Quantum Information Theory
Lecture 12

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The partial trace

The partial trace of an operator O with respect to system B is an operator on A:

$$\mathrm{tr}_B(O) = \sum_k \langle f_k|O|f_k\rangle = \sum_{jl} O_{jk,lk}|e_j\rangle\langle e_l|.$$

The partial trace on system B is a linear map from the set of joint operators to the set of operators on A. The partial trace of O with respect to system A is similarly defined as

$$\mathrm{tr}_A(O) = \sum_j \langle e_j|O|e_j\rangle = \sum_{km} O_{jk,jm}|f_k\rangle\langle f_m|.$$

The complete trace can be obtained by doing two partial traces

$$\mathrm{tr}(O) = \sum_{j,k} O_{jk,jk} = \mathrm{tr}_A(\,\mathrm{tr}_B(O)) = \mathrm{tr}_B(\,\mathrm{tr}_A(O)).$$

In the same way as the complete trace, the partial trace is also independent of the orthogonal basis used to compute it. We also have

$$\mathrm{tr}_B(A\otimes B) = A\,\mathrm{tr}(B), \quad \mathrm{tr}_B((A\otimes \mathbb{1})O) = A\,\mathrm{tr}_B(O), \quad \mathrm{tr}_B(O(A\otimes \mathbb{1})) = \mathrm{tr}_B(O)\,A.$$

$$\mathrm{tr}_B(OO') = \sum_k \langle f_k|OO'|f_k\rangle = \sum_{k,m}\langle f_k|O|f_m\rangle\langle f_m|O'|f_k\rangle \neq \mathrm{tr}_B(O'O).$$

When dealing with the complete trace we can switch the order of operators since the matrix elements that appear while computing the trace are just numbers. In the case of the partial trace the sandwiches that appear are operators on subsystem A and in general they do not commute and hence we cannot, in general, switch the order of operators inside a partial trace. It still is true that

$$\mathrm{tr}_B((\mathbb{1}\otimes B)\,O) = \mathrm{tr}_B(O\,(\mathbb{1}\otimes B)),$$

because in this case the partial matrix element $\langle f_k|\mathbb{1}_A\otimes B|f_m\rangle = \mathbb{1}_A\,\langle f_k|B|f_k\rangle$ is a multiple of the identity operator on system A and therefore it will commute with the operator $\langle f_k|O|f_k\rangle$. Similarly we can show that

$$\begin{aligned}\mathrm{tr}_B(O\,(\mathbb{1}_A\otimes|\phi_B\rangle\langle\xi_B|)) &= \sum_{k,m}\mathbb{1}_A\,\langle f_k|O|f_m\rangle\langle f_m|\phi_B\rangle\langle\xi_B|f_k\rangle \\ &= \sum_{k,m}\mathbb{1}_A\,\langle\xi_B|f_k\rangle\langle f_k|O|f_m\rangle\langle f_m|\phi_B\rangle \\ &= \mathbb{1}_A\,\langle\xi_B|O|\phi_B\rangle = \langle\xi_B|O|\phi_B\rangle,\end{aligned}$$

which means that a partial trace on B does indeed make an outer product on B into a partial matrix element with respect to B. One needs to keep track of which one is an operator on A and which one is a complex number when doing these manipulations. In this case, $\langle \xi_B|O|\phi_B\rangle$ is an operator on system A while $\langle f_m|\phi_B\rangle$ and $\langle \xi_B|f_k\rangle$ are complex numbers.

The main use we have for the partial trace is to compute the marginal density operator for a subsystem of a composite system. If the composite system has a state ρ_{AB} then a measurement in the basis $|e_j\rangle$ on system A yields result j with probability

$$p_j = \sum_k p_{jk} = \sum_k \langle e_j, f_k|\rho_{AB}|e_j, f_k\rangle.$$

In writing the above expression we assume that in addition to the measurement in the basis $|e_j\rangle$ on system A we are measuring system B in an arbitrary basis $|f_k\rangle$. To find the probability for a particular outcome for the measurement on A we compute the marginal distribution by summing over all the results of measurement on B in accordance with the rules of classical probabilities. The probability for result j can be put in the form

$$p_j = \sum_k \langle e_j, f_k|\rho_{AB}|e_j, f_k\rangle = \langle e_j|\Big(\sum_k \langle f_k|\rho_{AB}|f_k\rangle\Big)|e_j\rangle = \langle e_j|\,\mathrm{tr}_B(\rho_{AB})|e_j\rangle = \langle e_j|\rho_A|e_j\rangle,$$

where we have defined

$$\rho_A = \mathrm{tr}_B(\rho_{AB}).$$

The operator ρ_A is called the marginal density operator of A since as far as subsystem A is concerned all probabilities are computed as though ρ_A is the state of subsystem A. We can put p_j in other useful forms as follows:

$$\begin{aligned}
p_j &= \sum_k \mathrm{tr}\big(\rho_{AB}|e_j, f_k\rangle\langle e_j, f_k|\big) \\
&= \sum_k \mathrm{tr}\big(\rho_{AB}|e_j\rangle\langle e_j| \otimes |f_k\rangle\langle f_k|\big) \\
&= \mathrm{tr}\Bigg(\rho_{AB}|e_j\rangle\langle e_j| \otimes \Big(\sum_k |f_k\rangle\langle f_k|\Big)\Bigg) \\
&= \mathrm{tr}(\rho_{AB}(P_j \otimes \mathbb{1}_B)) \\
&= \mathrm{tr}_A\big(\,\mathrm{tr}_B(\rho_{AB}(P_j \otimes \mathbb{1}_B))\big) \\
&= \mathrm{tr}_A\big(\,\mathrm{tr}_B(\rho_{AB})P_j\big) \\
&= \mathrm{tr}_A\big(\rho_A P_j\big).
\end{aligned}$$

Note that in the middle when we are working with the composite state ρ_{AB}, the projection operator that goes with the measurement is $P_j \otimes \mathbb{1}_B$ which not a rank 1 projector but instead has rank d_B. This corresponds to the fact that the result j is degenerate, with d_B different possibilities for system B.

Let us now consider an example. Consider a two qubit pure state,

$$\Psi_{AB}\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle.$$

The corresponding composite density operator is

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}| = \cos^2\theta|00\rangle\langle 00| + \sin^2\theta|11\rangle\langle 11| + \cos\theta\sin\theta(|00\rangle\langle 11| + |11\rangle\langle 00|).$$

A measurement of

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|,$$

only on system A yields the result $+1$ with probability,

$$\begin{aligned} p_{+1} &= \operatorname{tr}(\rho_{AB}\, P_{e_z} \otimes \mathbb{1}_B) \\ &= \langle\Psi_{AB}|P_{e_z} \otimes \mathbb{1}_B|\Psi_{AB}\rangle \\ &= \langle\Psi_{AB}|\big(|0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)\big)|\Psi_{AB}\rangle \\ &= \langle\Psi_{AB}|\big(|00\rangle\langle 00| + |01\rangle\langle 01|\big)|\Psi_{AB}\rangle \\ &= \cos^2\theta. \end{aligned}$$

We can get the same result by computing the marginal density operator ρ_A first,

$$\begin{aligned} \rho_A &= \operatorname{tr}_B(\rho_{AB}) \\ &= \cos^2\theta\,\operatorname{tr}_B(|00\rangle\langle 00|) + \sin^2\theta\,\operatorname{tr}_B(|11\rangle\langle 11|) \\ &\qquad\qquad + \cos\theta\sin\theta\big(\operatorname{tr}_B(|00\rangle\langle 11|) + \operatorname{tr}_B(|11\rangle\langle 00|)\big) \\ &= \cos^2\theta|0\rangle\langle 0|\langle 0|0\rangle + \sin^2\theta|1\rangle\langle 1|\langle 1|1\rangle + \cos\theta\sin\theta(|0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle) \\ &= \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|, \end{aligned}$$

from which we get,

$$p_{+1} = \operatorname{tr}_A(\rho_A P_{e_z}) = \operatorname{tr}_A(\rho_A|0\rangle\langle 0|) = \langle 0|\big(\cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|\big)|0\rangle = \cos^2\theta.$$