PHY 4105: Quantum Information Theory
Lecture 25

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Quantum information

To quantify various aspects of the classical information in random variables X and Y taking values x_j and y_j respectively with probabilities $p(x_j)$ and $p(y_j)$ we had defined the following quantities:

1. Shannon entropy:

$$H(\vec{p}) = H(X) = -\sum_j p(x_j) \log p(x_j).$$

   The Shannon entropy was connected to out ignorance about the random variable X, we used it for understanding typical sequences and block coding etc.

2. Relative entropy

$$H(\vec{p}\|\vec{q}) = \sum_j p_j \log \frac{p_j}{q_j} = -H(\vec{p}) - \sum_j p_j \log q_j \geq 0.$$

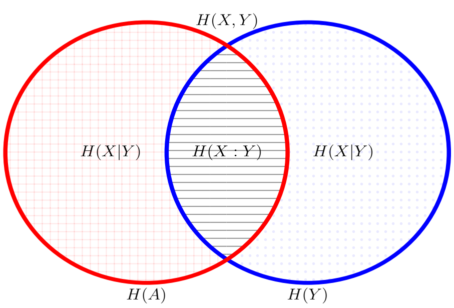
3. Conditional entropy

$$H(X|Y) = \sum_y p(y) \Big(-\sum_x p(x|y) \log p(x|y) \Big) = -\sum_{x,y} p(x,y) \log p(x|y).$$

4. Mutual information

$$H(X:Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y).$$

The properties of Shannon entropy for two variables is described best by the Venn diagram below:



We can now ask the question how we quantify the quantum information contained in a state ρ. We can define an analogue of the Shannon entropy as follows. We start with the eigen-decomposition of the state as

$$\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|,$$

and define the vonNeumann entropy of the state as

$$S(\rho) = H(\vec{\lambda}) = -\sum_j \lambda_j \log \lambda_j = -\mathrm{tr}(\rho \log \rho).$$

We have the following properties for the vonNeumann entropy:

1. vonNeumann entropy is positive semi-definite,

$$0 \leq S(\rho) \leq \log D.$$

   The first inequality is saturated ($S(\rho) = 0$) for a pure state which means that if we know that a system is in a particular pure state then there is no further ignorance about it. The maximally mixed state in D dimensions has the maximal entropy.

2. The ODOP inequality: If we do a complete set of one dimensional projective measurements on a quantum system yielding results j with probability $q_j = \mathrm{tr}(\rho|f_j\rangle\langle f_j|)$, then

$$H(\vec{q}) \geq H(\vec{\lambda}) = S(\rho).$$

   So a complete set of measurements can put an upper bound on the vonNeumann entropy.

3. Concavity:

$$S(\mu\rho_1 + (1-\mu)\rho_2) \geq \mu S(\rho_1) + (1-\mu)S(\rho_2), \qquad 0 < \mu < 1.$$

Analogous to the classical relative entropy, we can define a quantum relative entropy as

$$S(\rho\|\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) = -S(\rho) - \mathrm{tr}(\rho \log \sigma).$$

The relative entropy is like a distance between two density matrices but it is not symmetric.

If we have two systems, we have a joint density matrix ρ_{AB} and sub-system (partial trace) density matrices ρ_A and ρ_B. We can define vonNeumann entropies for all there. If ρ_{AB} is a pure state then $S(\rho_{AB}) = 0$ and $S(\rho_A) = S(\rho_B)$. We also have subadditivity:

$$S(A,B) \leq S(A) + S(B),$$

with equality when $\rho_{AB} = \rho_A \otimes \rho_B$. There is also a triangle inequality (Araki-Lieb),

$$S(A,B) \geq |S(A) - S(B)|.$$

We can have quantum conditional entropies defined as

$$S(A|B) = S(A,B) - S(B), \qquad S(B|A) = S(A,B) - S(B).$$

It is easy to see that the conditional entropies as defined can be negative and so we have to think a bit harder about what it means to "know" the sub-system. The quantum mutual information, however, is positive semi-definite

$$S(A:B) = S(A) + S(B) - S(A,B).$$

Holevo bound: This gives us how much classical information can be sent down a noiseless quantum channel. The only noise that comes here is because of the quantum measurement at the end. At the input side, Alice sends ρ_x with probability p_x. At the output end Bob measure a POVM E_y,

$$p_{y|x} = \mathrm{tr}(E_y \rho_x).$$

The Holevo bound on the mutual information between the random variables X and Y (What was sent by Alice and what Bob infers) is

$$H(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x) = \chi,$$

where χ is referred to as the Hoelvo quantity. The maximum value of $H(X:Y)$ is called the accessible information.