PHY 4105: Quantum Information Theory
Lecture 6

Anil Shaji

School of Physics, IISER Thiruvananthapuram

(Dated: August 20, 2013)

A Brief Review of Quantum Mechanics

A complete description of the state of a quantum system is given by a vector in the Hilbert space associated with the system. The state vector - often also called the wave function - may contain a variety of information about the system, all packaged in a form that respects the uncertainty principle. For instance

$$|\Psi_{s,c,\vec{k}}(\vec{x})\rangle = A \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \otimes \cdots .$$

The Hilbert space is a complex vector space endowed with an inner product,

$$(|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle.$$

The 'Bra' vector $|\phi\rangle$ is defined as a linear functional on $|\psi\rangle$ that maps the kets on to $\mathbb{C}$.

1. The inner product is linear in $|\psi\rangle$:

$$(|\phi\rangle, a|\psi_1\rangle + b|\psi_2\rangle) = a(|\phi\rangle, |\psi_1\rangle) + b(|\phi\rangle, |\psi_2\rangle).$$

2. Complex symmetric:

$$(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*.$$

3. Anti-linearity in $|\phi\rangle$,

$$(a|\phi_1\rangle + b|\phi_2\rangle, |\psi\rangle) = a^*(|\phi_1\rangle, |\psi\rangle) + b^*(|\phi_2\rangle, |\psi\rangle).$$

4. $(|\psi\rangle, |\psi\rangle) \geq 0$ with equality if and only if $|\psi\rangle = 0$.

We can identify a basis $|e_j\rangle = |j\rangle$, $j = 1, \ldots D$ with $\langle e_j|e_k\rangle = \delta_{jk}$ and expand any vector in terms of the basis as

$$|\psi\rangle = \sum_j c_j|e_j\rangle = \sum_j \langle e_j|\psi\rangle|e_j\rangle, \qquad c_j = \langle e_j|\psi\rangle.$$

Similarly

$$|\phi\rangle = \sum_j d_j|e_j\rangle = \sum_j |e_j\rangle\langle e_j|\phi\rangle,$$

and

$$\langle\phi|\psi\rangle = \sum_j d_j^* c_j = \sum_j \langle\phi|e_j\rangle\langle e_j|\psi\rangle.$$

The expansion coefficients of the vectors furnish a representation of the Hilbert space vectors and their duals as column/row vectors:

$$|\psi\rangle \leftrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}, \qquad \langle\phi| \leftrightarrow (d_1^* \ldots d_D^*), \qquad \langle\phi|\psi\rangle = (d_1^* \ldots d_D^*) \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}.$$

Hilbert space vectors satisfy the Shwarz inequality:

$$|\langle\phi|\psi\rangle| \leq \langle\phi|\phi\rangle^{1/2}\langle\psi|\psi\rangle^{1/2}.$$

Proof: We start with defining a new Hilbert space vector,

$$|\xi\rangle \equiv |\psi\rangle\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}.$$

Let

$$|\chi\rangle = |\phi\rangle - |\xi\rangle, \qquad \Rightarrow \qquad |\phi\rangle = |\chi\rangle + |\xi\rangle.$$

The Shwarz inequality is essentially the statement that the vector $|\chi\rangle$ has non-zero length ($\langle\chi|\chi\rangle \geq 0$). Note that as of now we have not required that $|\phi\rangle$ and $|\chi\rangle$ be normalized vectors.

$$\langle\xi|\chi\rangle = \langle\xi|\phi\rangle - \langle\xi|\xi\rangle = \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\phi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{|\langle\psi|\psi\rangle|^2}\langle\psi|\psi\rangle = 0.$$

So

$$\langle\phi|\phi\rangle = \langle\chi|\chi\rangle + \langle\xi|\xi\rangle \geq \langle\xi|\xi\rangle = \frac{|\langle\psi|\phi\rangle|^2}{|\langle\psi|\psi\rangle|^2}\langle\psi|\psi\rangle = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle},$$

$$\Rightarrow \langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq |\langle\phi|\psi\rangle|^2.$$

Linear Operators

We have described quantum kinematics using vectors in Hilbert space. Now we have to establish the mathematical tools for describing changes to the state of quantum systems. Due to the linearity of quantum mechanics, linear operators on Hilbert space vectors fill this role. Linear operators act as

$$A(a|\psi\rangle + b|\phi\rangle) = aA|\psi\rangle + bA|\phi\rangle.$$

Choosing a basis $|e_j\rangle$, we can have a matrix representation of a linear operator as

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1D} \\ A_{21} & A_{22} & \cdots & A_{2D} \\ \vdots & & \vdots & \\ A_{D1} & A_{D2} & \cdots & A_{DD} \end{pmatrix}, \qquad A_{jk} = \langle e_j|A|e_k\rangle.$$

In terms of the matrix representation, we have

$$A|\psi\rangle = \sum_j |e_j\rangle\langle e_j A|\psi\rangle = \sum_{j,k} |e_j\rangle\langle e_j|A|e_k\rangle\langle e_k|\psi\rangle. = \sum_{j,k} |e_j\rangle A_{jk} c_k.$$

We can equivalently represent an operator using the outer-product representation as

$$A = \sum_{jk} |e_j\rangle A_{jk}\langle e_k| = \sum_j |e_j\rangle\langle e_j|A|e_k\rangle\langle e_k|,$$

Here we have used the resolution of the identity operator as

$$\mathbb{1} = \sum_j |e_j\rangle\langle e_j|.$$

Product of two operators:

$$\begin{aligned} AB &= \sum_{jklm} |e_j\rangle\langle e_j|A|e_l\rangle\langle e_l|e_m\rangle\langle e_m|B|e_k\rangle\langle e_k| \\ &= \sum_{jkl} |e_j\rangle\langle e_j|A|e_l\rangle\langle e_l|B|e_k\rangle\langle e_k| \\ &= \sum_{jk} \Big(\sum_l A_{jl}B_{lk}\Big)|e_j\rangle\langle e_k|. \end{aligned}$$

Adjoint (Hermitian conjugate) operator: The adjoint of an operator is defined by the relation

$$(|\phi\rangle, A|\psi\rangle) = (A^\dagger|\phi\rangle, |\psi\rangle) = (|\psi\rangle, A^\dagger|\phi\rangle)^*.$$

In the Dirac notation we have

$$\langle\phi|A|\psi\rangle = \langle\psi|A^\dagger|\phi\rangle^*.$$

In terms of dual vectors, $\langle\phi|A^\dagger$ is the dual of $A|\phi\rangle$ and $\langle\phi|A$ is the dual of $A^\dagger|\phi\rangle$. Some of the properties of the adjoint are listed below:

1. Adjoint is anti linear: $(aA + bB)^\dagger = a^*A^\dagger + b^*B^\dagger$
2. $(A^\dagger)_{jk} = \langle e_j A^\dagger|e_k\rangle = \langle e_k|A|e_j\rangle^* = A^*_{kj} \quad \Rightarrow \quad (A^\dagger)^\dagger = A.$
3. $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi| \quad \Rightarrow \quad A^\dagger = \sum_{jk} |e_j\rangle\langle e_k|A^*_{kj}.$
4. $(AB)^\dagger = B^\dagger A^\dagger.$

A projection operator projects on to a direction in Hilbert space:

$$P_\psi = |\psi\rangle\langle\psi|, \quad \langle\psi|\psi\rangle = 1.$$

$$P_\psi|\phi\rangle = |\psi\rangle\langle\psi|\phi\rangle.$$

We can also define a projector on to a multidimensional subspace of the Hilbert space spanned by the vectors $|e_j\rangle$, $j = 1, \ldots, D$ as

$$S = \sum_{j=1}^{D} |e_j\rangle\langle e_j|.$$

The operator S is the unit operator in the subspace.

An operator has a spectral decomposition or an eigenvalue decomposition if it can be written in the form

$$A = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_j \lambda_j P_j = \sum_\lambda \lambda P_\lambda, \qquad \sum_j P_j = \sum_\lambda P_\lambda = \mathbb{1}.$$

The vectors $|e_j\rangle$ are the eigenvectors of A,

$$A|e_j\rangle = \lambda_j |e_j\rangle.$$

Multiple eigenvectors can correspond to the same eigenvalue λ. These are said to be degenerate eigenvectors. We can group the degenerate eigenvectors into subspaces or eigenspaces S. Eigenvectors corresponding to different eigenvalues are orthogonal to each other. $P_\lambda = \sum_{\lambda_j=\lambda} |e_j\rangle\langle e_j|$ is the projector on to the subspace spanned by all the eigenvectors with eigenvalue λ.

The support of an operator is the subspace S spanned by all its eigenvectors corresponding to non-zero eigenvalues. The null space K of the operator is the complement of the support and is spanned by eigenvectors with zero eigenvalues. The two subspaces are mutually orthogonal.

The commutator of two operators is defined as

$$[A, B] = AB - BA.$$

A Normal operator M is one such that $[M, M^\dagger] = 0$.

Theorem: An operator has a spectral decomposition if and only if it is normal.

Hermitian operators are normal operators whose eigenvalues are all real. As can be easily seen from the spectral decomposition, Hermitian operators are adjoints of themselves (self-adjoint) $H = H^\dagger$

Another useful class of operators are normal operators whose eigenvalues are phases of the kind $e^{i\phi}$. These are the unitary operators and they satisfy:

$$UU^\dagger = U^\dagger U = \mathbb{1}.$$

Unitary operators preserve inner products, take orthonormal bases to orthonormal bases and rows and columns of unitary operators are themselves orthogonal vectors.

A projection operator is a Hermitian operator whose eigenvalues are either 0 or 1. Alternatively, a Hermitian operator P is a projector if $P^2 = P$.