PHY 4105: Quantum Information Theory
Lecture 7

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The commutator of two operators is defined as

$$[A,\, B] = AB - BA.$$

A Normal operator M is one such that $[M,\, M^\dagger] = 0$.
Theorem: An operator has a spectral decomposition if and only if it is normal.

Hermitian operators are normal operators whose eigenvalues are all real. As can be easily seen from the spectral decomposition, Hermitian operators are adjoints of themselves (self-adjoint) $H = H^\dagger$

Another useful class of operators are normal operators whose eigenvalues are phases of the kind $e^{i\phi}$. These are the unitary operators and they satisfy:

$$UU^\dagger = U^\dagger U = \mathbb{1}.$$

Unitary operators preserve inner products, take orthonormal bases to orthonormal bases and rows and columns of unitary operators are themselves orthogonal vectors.

A projection operator is a Hermitian operator whose eigenvalues are either 0 or 1. Alternatively, a Hermitian operator P is a projector if $P^2 = P$.

Normal operators have the additional desirable quality that any function on complex variables, $f : \mathbb{C} \to \mathbb{C}$ can be extended to a function on the normal operators by

$$f(A) = f(\sum_j \lambda_j |e_j\rangle\langle e_j|) = \sum_j f(\lambda_j)|e_j\rangle\langle e_j|.$$

As an example, consider the unitary operator,

$$U = \sum_j e^{i\phi_j}|e_j\rangle\langle e_j|.$$

We can now construct a Hermitian operator,

$$H = \sum_j \phi_j |e_j\rangle\langle e_j|,$$

such that

$$U = \sum_j e^{i\phi_j}|e_j\rangle\langle e_j| = e^{iH}.$$

In fact any unitary can be written as a the exponential of a Hermitian operator.

Let us now introduce the trace operation on an operator as

$$\mathrm{tr}(A) \equiv \sum_j \langle e_j|A|e_j\rangle.$$

1. The trace is a linear operation

$$\mathrm{tr}(aA + bB) = a\,\mathrm{tr}(A) + b\,\mathrm{tr}(B).$$

2. Trace is basis independent despite the definition having an explicit choice of basis in it:

$$\begin{aligned}\sum_j \langle e_j|A|e_j\rangle &= \sum_{jk} \langle e_j|f_k\rangle\langle f_k|A|e_j\rangle \\ &= \sum_{jk} \langle f_k|A|e_j\rangle\langle e_j|f_k\rangle \\ &= \sum_k \langle f_k|A|f_k\rangle,\end{aligned}$$

   where we have first inserted a resolution of the identity and then removed it in the next step.

3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

4. $\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$: cyclic property

5. $\mathrm{tr}(|\psi\rangle\langle\phi|) = \sum_j \langle e_j|\psi\rangle\langle\phi|e_j\rangle = \sum_j \langle\phi|e_j\rangle\langle e_j|\psi\rangle = \langle\phi|\psi\rangle$.

6. $\mathrm{tr}(A|\psi\rangle\langle\phi|) = \mathrm{tr}(|\psi\rangle\langle\phi|A) = \langle\phi|A|\psi\rangle$.

The last two properties say that the trace operation turns an outer product into a sandwich.

Theorem 1. *Simultaneous eigenvectors theorem: Two normal operators have simultaneous eigenvectors if an only if they commute.*

Proof. If A and B have simultaneous eigenvectors:

$$A = \sum_j \lambda_j |e_j\rangle\langle e_j|, \qquad B = \sum_j \mu_j |e_j\rangle\langle e_j|,$$

Then

$$[A,\, B] = \sum_{jk} \lambda_j \mu_k [|e_j\rangle\langle e_j|,\, |e_k\rangle\langle e_k|] = 0.$$

To prove the reverse, we have $[A,\, B] = 0$. Let A have an eigendecompostion

$$A = \sum_\lambda \lambda P_\lambda,$$

with P_λ projecting on to degenerate eigen subspaces S_λ of A. For any vector $|e\rangle$ in S_λ ($A|e\rangle = \lambda|e\rangle$), we have

$$A(B|e\rangle) = B(A|e\rangle) = \lambda B|e\rangle.$$

This means that $B|e\rangle$ is also an eigenvector of A with the eigenvalue λ. This means that B maps each eigen subspace of A into itself. So we can write the operator B as

$$B = \sum_\lambda B_\lambda,$$

where B_λ acts only within the corresponding eigen subspace S_λ. Diagonalizing the B_λs within each subspace S_λ yields the eigenvectors of B which are also eigenvectors of A.

Note that if B eigenvectors in different S_λ are degenerate then we can superpose them to create eigenvectors of B which do not lie in any of the S_λs. □

Quantum mechanics: Axioms

Now that we have the mathematical tools to describe quantum states and changes to those states, let us connect it to observations and experiments by formulating the axioms of quantum mechanics in terms of Hilbert space vectors, operators etc.

1. A state of a quantum system is a *ray* in Hilbert space. A ray is the collection of state vectors $a|\psi\rangle$, $a \in \mathbb{C}$. The set of rays is called a projective Hilbert space.

   (a) We usually normalize the rays so that $\langle\psi|\psi\rangle = 1$. However still $e^{i\theta}|\psi\rangle$ is the same state as $|\psi\rangle$. The overall (global) phase on the state does not matter and has no observable consequences. A normalized vector is called a state vectors.

   (b) The phase ambiguity can be removed by identifying the state of a quantum system not with a ray in Hilbert space but with a one dimensional projector $|\psi\rangle\langle\psi|$. This leads to the density matrix representation of the states which we will look into in detail later.

2. An observable is a Hermitian linear operators,

$$A = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_\lambda \lambda P_\lambda.$$

   The result of a measurement of A is one of the eigenvalues λ. When the system is in state $|\psi\rangle$ the probability of getting the measurement result as λ is

$$p(\lambda) = \langle\psi|P_\lambda|\psi\rangle = \sum_{\lambda_j=\lambda} |\langle e_j|\psi\rangle|^2 = \operatorname{tr}(P_\lambda|\psi\rangle\langle\psi|).$$

   $|\langle e_j|\psi\rangle|^2$ is the probability for eigenvector $|e_j\rangle$ and $\langle e_j|\psi\rangle$ is the corresponding probability amplitude.

   The eigenvalues label the measurement results while the probabilities for each outcome depend on the eigenvectors.

3. A measurement with result $|e_j\rangle$ leaves the system in the state $|e_j\rangle$ (collapse of the wave function). So after the measurement the state is

$$|\psi\rangle \to \frac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}} = \frac{P_j|\psi\rangle}{p_j}.$$

4. The system has a Hamiltonian H. If the system is closed or isolated from everything else in the universe then its state vector evolves according the the Schrödinger equation,

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle \;\Rightarrow\; |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle.$$

   The finite time evolution operator is given by the unitary operator $U(t)$.

5. Expectation values

$$\langle A\rangle = \sum_{\lambda} p(\lambda) = \langle\psi|\sum_{\lambda}\lambda P_{\lambda}|\psi\rangle = \langle\psi|A|\psi\rangle.$$

As we proceed most of these will have to be generalized especially in the context of dealing with quantum systems that are interacting in uncontrollable ways with their respective environments. Then we will have to learn about density operators for representing states, POVMs representing measurements, quantum operations and dynamical maps for evolution etc.