PHY 4105: Quantum Information Theory
Lecture 9

Anil Shaji
School of Physics, IISER Thiruvananthapuram
(Dated: August 28, 2013)

Observables on qubits

If a qubit is in a state $P_{\vec{n}} = |\vec{n}\rangle\langle\vec{n}|$, then a measurement of the Hermitian operator $\vec{m}\cdot\vec{\sigma}$ will yield the result $+1$ ($|\vec{m}\rangle$) with probability,

$$|\langle\vec{m}|\vec{n}\rangle|^2 = \mathrm{tr}(P_{\vec{n}}P_{\vec{m}}) = \frac{1}{2}(1+\vec{m}\cdot\vec{n}),$$

while the measurement will yield the result -1 ($|-\vec{m}\rangle$) with probability,

$$|\langle-\vec{m}|\vec{n}\rangle|^2 = \mathrm{tr}(P_{\vec{n}}P_{-\vec{m}}) = \frac{1}{2}(1-\vec{m}\cdot\vec{n}),$$

Unitary dynamics

Any unitary operator acting on a single qubit can be written as

$$U = e^{i(\delta\mathbb{1}-\vec{n}\cdot\vec{\sigma}\theta/2)} = e^{i\delta}e^{-i\vec{n}\cdot\vec{\sigma}\theta/2}.$$

The first factor of $e^{i\delta}$ is just an overall global phase. The second factor,

$$U_R = e^{-i\vec{n}\cdot\vec{\sigma}\theta/2} = e^{-i\vec{n}\cdot\vec{S}\theta},$$

is the canonical rotation operator on a state in two dimensional Hilbert space corresponding to a rotation through angle θ about the axis defined by $\vec{n}$. We can show that

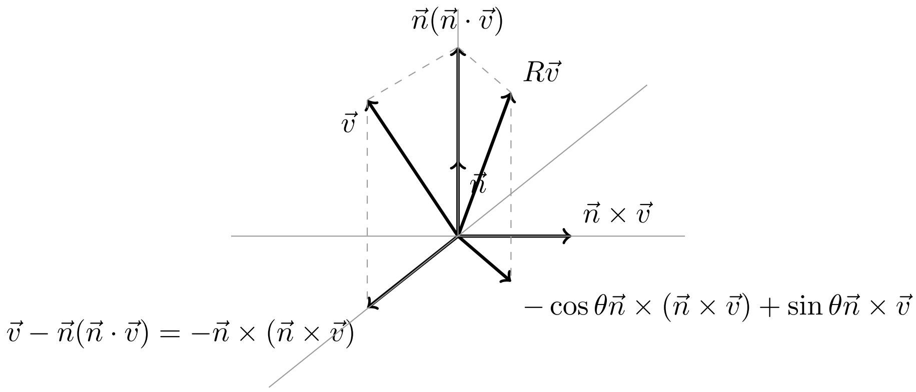
$$U_R^\dagger\vec{\sigma}U_R = R_{\vec{n}}(\theta)\vec{\sigma},$$

where $R_{\vec{n}}(\theta)$ is the rotation operator in real three dimensional space corresponding of a rotation through an angle θ about the axis $\vec{n}$. $R_{\vec{n}}(\theta)$ is an orthogonal matrix with $R^TR=\mathbb{1}$. Proving this can be done in the most straightforward way possible, starting from each one of the three Pauli matrices and showing that

$$U_R^\dagger\sigma_jU_R = R_{jk}\sigma_k, \qquad U_R\sigma_jU_R^\dagger = \sigma_kR_{kj}.$$

We will omit the details here since they are rather long and tedious but we use (as seen from the figure below)

$$\begin{aligned} R\vec{v} &= \vec{n}(\vec{n}\cdot\vec{v}) - \cos\theta\vec{n}\times(\vec{n}\times\vec{v}) + \sin\theta\vec{n}\times\vec{v} \\ &= \vec{v}\cos\theta + (1-\cos\theta)\vec{n}(\vec{n}\cdot\vec{v}) + \sin\theta\vec{n}\times\vec{v}. \end{aligned}$$



As a consequence we have

$$U_R\vec{\sigma}\cdot\vec{m}U_R^\dagger = R^T\vec{\sigma}\cdot\vec{m} = \vec{\sigma}\cdot R\vec{m}.$$

$$(\vec{\sigma}\cdot R\vec{m})U_R|\vec{m}\rangle = U_R\vec{\sigma}\cdot\vec{m}|\vec{m}\rangle = U_R|\vec{m}\rangle \quad \Rightarrow U_R|\vec{m}\rangle = e^{i\phi(R,\vec{m})}|R\vec{m}\rangle.$$

This means that unitary dynamics rotates states on the Bloch sphere.

As an example consider the following Hamiltonian for a spin-1/2 system:

$$H = -\vec{\mu}\cdot\vec{B} = \frac{1}{2}\gamma B\hbar Z.$$

This is the Hamiltonian for a spin with magnetic moment $\vec{\mu}$ in a magnetic field $\vec{B}$. We choose $\vec{B} = B\hat{e}_z$ and $\vec{\mu} = -\gamma\vec{S} = -\frac{1}{2}\gamma\hbar\vec{\sigma}$ to get the Hamiltonian above.

The Unitary time evolution operator is

$$U(t) = e^{-iHt/\hbar} = e^{-i(\gamma Bt/2)Z}.$$

This corresponds to the rotation about $\hat{e}_z$ of states of the spin through an angle γBt at time t (Rabi flopping etc).

As far as qubits are concerned, the structure of the σ_z matrix tells us why by convention we choose to label the top of the Bloch sphere as $|0\rangle$ which it suggests the ground state or lower spin state. That is because we want to write

$$\sigma_z|x\rangle = (-1)^x|x\rangle.$$

We will wrap up our discussion on qubits and unitaries on qubits by noting that

$$U_R = e^{-i\vec{n}\cdot\vec{\sigma}\,\theta/2} = e^{-i\theta/2}|\vec{n}\rangle\langle\vec{n}| + e^{i\theta/2}|-\vec{n}\rangle\langle-\vec{n}|.$$

This means that

$$\det U_R = e^{-i\theta/2}e^{i\theta/2} = 1.$$

But

$$\det U = e^{2i\delta}, \qquad U = e^{i\delta}U_R.$$

Requiring U to have unit determinant, $U \in SU(2)$ means that $\delta = 0$ or π. We don't need the π case which changes the sign of U since advancing θ by 2π also changes the sign of U (double-valued representation of SU(2)).